MATH 111
EXAM 01
SOLUTIONS (V1)

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1. Definitions

1 (3 Points)**.** Fill in the blanks with the correct factorizations.

(a) $A^2 - B^2 = (A - B)(A + B)$.

(b) $A^2 + 2AB + B^2 = (A + B)^2$.

(c) $A^2 - 2AB + B^2 = (A - B)^2$.

2 (6 Points)**.** Let a, b be non-zero real numbers and m, n integers. Fill in the blanks

(1) $a^0 = 1$

(2) $a^{-n} = \dfrac{1}{a^n}$

(3) $a^m \cdot a^n = a^{m+n}$

(4) $\dfrac{a^m}{a^n} = a^{m-n}$

(5) $(a \cdot b)^n = a^n b^n$

(6) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

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3 (2 Points)**.** Given a Quadratic Equation, $ax^2 + bx + c = 0$, the solutions are given by the Quadratic Formula. State the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

4 (3 Points)**.** Fill in the blanks:

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

5 (3 Points)**.** Fill in the blanks:

An equation in variables x and y defines the variable y as function of the variable x if

each value of x corresponds to exactly one value of y.

6 (3 Points)**.** Let $f(x)$ be a function of the variable x, and assume that $a \leq b$ are real numbers. State the formula for the net change between the inputs $x = a$ and $x = b$.

The net change is given by the formula

$$f(b) - f(a).$$

2. Problems

7 (16 Points)**.** Consider the equation

$$x^2 + 2y = 6.$$

(a) Does this equation define y as function of x? *Briefly* justify why or why not. If it does, then give the value of y when $x = 2$.

Solution. This equation does define y as a function of x. Subtracting x^2 from both sides we obtain

$$2y = 6 - x^2.$$

Dividing both sides by 2 we obtain

$$y = \frac{6 - x^2}{2}$$

and this is clearly a function. When $x = 2$,

$$y = \frac{6 - 2^2}{2} = \frac{6 - 4}{2} = \frac{2}{2} = 1.$$

(b) Does this equation define x as function of y? *Briefly* justify why or why not. If it does, then give the value of x when $y = 2$.

Solution. This equation does *not* define x as a function of y. When $y = 2$ we have the equation

$$6 = x^2 + 2(2) = x^2 + 4.$$

Subtracting 4 from both sides yields

$$x^2 = 6 - 4 = 2$$

so there are two choices for value of x: either $\sqrt{2}$ or $-\sqrt{2}$.

8 (16 Points)**.** Let $f(x) = x^2 - x + 1$. Compute the net change between $x = 1$ and $x = 4$.

Solution. The net change is given by

$$f(4) - f(1) = (4^2 - 4 + 4) - (1^2 - 1 + 1) = 16 - 1 = 15.$$

9 (16 Points)**.** Add the following rational expressions and simplify the result,

$$\frac{1}{x-3}+\frac{-6}{x^2-9}.$$

Solution. First, we must factor the denominators to determine the least common multiple. The denominator $x-3$ cannot be factored any further, but

$$x^2-9=(x-3)(x+3).$$

This tells us common denominator is x^2-9, so

$$\begin{aligned}\frac{1}{x-3}+\frac{-6}{x^2-9} &= \left(\frac{x+3}{x+3}\right)\frac{1}{x-3}-\frac{6}{x^2-9}.\\ &= \frac{x+3}{x^2-9}-\frac{6}{x^2-9}\\ &= \frac{x+3-6}{x^2-9}\\ &= \frac{x-3}{(x+3)(x-3)}\\ &= \frac{1}{x+3}.\end{aligned}$$

10 (16 Points)**.** Solve the equation

$$2x^2-6x-20=0$$

for x.

Solution. We first observe that we can factor a 2 out of the left-hand side to get

$$2(x^2-3x-10)=0.$$

By the Zero Factor Property, this says that either $2=0$ or $x^2-3x-10=0$; since the first equation is nonsense, it is equivalent to solve the equation

$$x^2-3x-10=0.$$

One either observes that

$$x^2-3x-10=(x-5)(x+2)=0$$

implies by the Zero Factor Property that the solutions are $x = 5$ and $x = -2$, or uses the Quadratic Formula

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)} \\ &= \frac{3 \pm \sqrt{9+40}}{2} \\ &= \frac{3 \pm \sqrt{49}}{2} \\ &= \frac{3 \pm 7}{2} \end{aligned}$$

to see that the the solutions are given by

$$x = \frac{3+7}{2} = \frac{10}{2} = 5$$

and

$$x = \frac{3-7}{2} = \frac{-4}{2} = -2.$$

11 (16 Points)**.** Find the domain of the function

$$f(x) = \sqrt{x^2 + 2x - 3}.$$

Express the solution using interval notation and graph the domain on the number line.

Solution. We first observe that for f to be well-defined at x, the value of $x^2 + 2x - 3$ must be non-negative. Hence finding the domain of f is equivalent to solving the quadratic inequality

$$0 \leq x^2 + 2x - 3.$$

We first solve the equation

$$x^2 + 2x - 3 = (x+3)(x-1) = 0$$

to see that equality is obtained whenever $x = 1$ or $x = -3$. To find the rest of the solutions to the inequality, we must check one number from each of the intervals $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$. We check the signs at $x = -4$, $x = 0$, and $x = 2$.

When $x = -4$, we have

$$(-4)^2 + 2(-4) - 3 = (-4+3)(-4-1) = (-1)(-5) = 5 > 0.$$

This tells us that $x^2 + 2x - 3$ is positive on the interval $(-\infty, -3)$.

When $x = 0$, we have

$$0^2 + 2(0) - 3 = -3 < 0.$$

This tells us that $x^2 + 2x - 3$ is negative on the interval $(-1, 3)$.

When $x = 2$, we have

$$(2)^2 + 2(2) - 3 = (2 + 3)(2 - 1) = (5)(1) = 5 > 0.$$

This tells us that $x^2 + 2x - 3$ is positive on the interval $(1, \infty)$

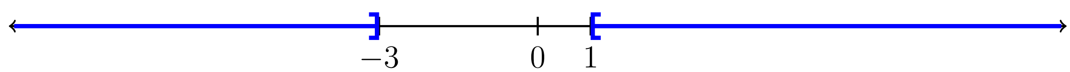
Combining this information, we have found that the inequality

$$0 \leq x^2 + 2x - 3$$

is true on the intervals $(-\infty, -3)$ and $(1, \infty)$. In interval notation, the solutions are therefore

$$(-\infty, -3) \cup (1, \infty).$$

We can represent this union of intervals on the number line as



12 (Bonus - 10 Points)**.** Consider the function

$$p(x) = x^2 + 5x + 6.$$

(a) Find two numbers, $a < b$, for which the net change of p from a to b is 6.

Solution. This question simply asks that we find two values $a < b$ such that

$$f(b) - f(a) = 6.$$

The easiest possible solution would be to find a and b such that $f(b) = 6$ and $f(a) = 0$. This is equivalent to solving the two equations

$$x^2 + 5x + 6 = 6 \text{ and } x^2 + 5x + 6 = 0$$

for x and identifying a solution to the first equation which is larger than some solution to the second equation.

To solve the first, we subtract 6 from both sides to obtain

$$x^2 + 5x = x(x + 5) = 0$$

and note that this means either $x = 0$ or $x = -5$. For the second, we can factor

$$x^2 + 5x + 6 = (x + 3)(x + 2) = 0$$

to see that either $x = -3$ or $x = -2$.

By some luck, we find two possible solutions. If we take $a = -2$, then we can take $b = 0$ since

$$f(0) - f(-2) = 6 - 0 = 6.$$

Similarly, if we take $a = -3$, then we can also take $b = 0$ since

$$f(0) - f(-3) = 6 - 0 = 0.$$

(b) Find two numbers, $c < d$, for which the net change of p from c to d is -6.
[Hint: Your method from part (a) should also give you these values.]

Solution. Again, we observe that the easiest thing we could do is find two numbers, $c < d$, such that $f(d) = 0$ and $f(c) = 6$ so that the net change is

$$f(d) - f(c) = 0 - 6 = -6.$$

Using our work from part (a), we can take $c = -5$ and either $d = -2$ or $d = -3$ since

$$f(-2) - f(-5) = 0 - 6 = -6$$

and

$$f(-3) - f(-5) = 0 - 6 = -6.$$

Remark 1. There may well be many more solutions. These are just the easiest such solutions.