MATH 111
EXAM 01
SOLUTIONS (V2)

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1. DEFINITIONS

1 (3 Points)**.** Fill in the blanks with the correct factorizations.

(a) $A^2 - B^2 = (A - B)(A + B)$.

(b) $A^2 + 2AB + B^2 = (A + B)^2$.

(c) $A^2 - 2AB + B^2 = (A - B)^2$.

2 (6 Points)**.** Let a, b be non-zero real numbers and m, n integers. Fill in the blanks

(1) $a^0 = 1$

(2) $a^{-n} = \dfrac{1}{a^n}$

(3) $a^m \cdot a^n = a^{m+n}$

(4) $\dfrac{a^m}{a^n} = a^{m-n}$

(5) $(a \cdot b)^n = a^n b^n$

(6) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

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3 (2 Points). Given a Quadratic Equation, $ax^2 + bx + c = 0$, the solutions are given by the Quadratic Formula. State the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

4 (3 Points). Fill in the blanks:

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

5 (3 Points). Fill in the blanks:

An equation in variables x and y defines the variable y as function of the variable x if

each value of x corresponds to exactly one value of y.

6 (3 Points). Let $f(x)$ be a function of the variable x, and assume that $a \leq b$ are real numbers. State the formula for the net change between the inputs $x = a$ and $x = b$.

The net change is given by the formula

$$f(b) - f(a).$$

2. PROBLEMS

7 (16 Points). Consider the equation

$$y^2 + 2x = 6.$$

(a) Does this equation define y as function of x? *Briefly* justify why or why not. If it does, then give the value of y when $x = 2$.

Solution. This equation does *not* define y as a function of x. When $x = 2$ we have the equation

$$6 = y^2 + 2(2) = y^2 + 4.$$

Subtracting 4 from both sides yields

$$y^2 = 6 - 4 = 2$$

so there are two choices for value of y: either $\sqrt{2}$ or $-\sqrt{2}$.

(b) Does this equation define x as function of y? *Briefly* justify why or why not. If it does, then give the value of x when $y = 2$.

Solution. This equation does define x as a function of y. Subtracting y^2 from both sides we obtain

$$2x = 6 - y^2.$$

Dividing both sides by 2 we obtain

$$x = \frac{6 - y^2}{2}$$

and this is clearly a function. When $y = 2$,

$$x = \frac{6 - 2^2}{2} = \frac{6 - 4}{2} = \frac{2}{2} = 1.$$

8 (16 Points). Let $f(x) = x^2 + x + 1$. Compute the net change between $x = 1$ and $x = 4$.

Solution. The net change is given by

$$f(4) + f(1) = (4^2 + 4 + 1) - (1^2 + 1 + 1) = 21 - 3 = 18.$$

9 (16 Points). Add the following rational expressions and simplify the result,

$$\frac{6}{9 - x^2} + \frac{-1}{3 + x}.$$

Solution. First, we must factor the denominators to determine the least common multiple. The denominator $3 + x$ cannot be factored any further, but

$$9 - x^2 = (3 - x)(3 + x).$$

This tells us common denominator is $9 - x^2$, so

$$
\begin{aligned}
\frac{6}{9 - x^2} + \frac{-1}{3 + x} &= \frac{6}{9 - x^2} + \left(\frac{3 - x}{3 - x}\right)\frac{-1}{3 + x}.\\
&= \frac{6}{9 - x^2} + \frac{-3 + x}{9 - x^2}\\
&= \frac{6 - 3 + x}{9 - x^2}\\
&= \frac{3 + x}{(3 + x)(3 - x)}\\
&= \frac{1}{3 - x}.
\end{aligned}
$$

10 (16 Points). Solve the equation

$$2x^2 - 2x - 24 = 0$$

for x.

Solution. We first observe that we can factor a 2 out of the left-hand side to get

$$2(x^2 - x - 12) = 0.$$

By the Zero Factor Property, this says that either $2 = 0$ or $x^2 - x - 12 = 0$; since the first equation is nonsense, it is equivalent to solve the equation

$$x^2 - x - 12 = 0.$$

One either observes that

$$x^2 - x - 12 = (x - 4)(x + 3) = 0$$

implies by the Zero Factor Property that the solutions are $x = 4$ and $x = -3$, or uses the Quadratic Formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 + 48}}{2}$$

$$= \frac{1 \pm \sqrt{49}}{2}$$

$$= \frac{1 \pm 7}{2}$$

to see that the the solutions are given by

$$x = \frac{1 + 7}{2} = \frac{8}{2} = 4$$

and

$$x = \frac{1 - 7}{2} = \frac{-6}{2} = -3.$$

11 (16 Points). Find the domain of the function

$$f(x) = \sqrt{x^2 - 2x - 3}.$$

Express the solution using interval notation and graph the domain on the number line.

Solution. We first observe that for f to be well-defined at x, the value of $x^2 - 2x - 3$ must be non-negative. Hence finding the domain of f is equivalent to solving the quadratic inequality

$$0 \le x^2 - 2x - 3.$$

We first solve the equation

$$x^2 - 2x - 3 = (x - 3)(x + 1) = 0$$

to see that equality is obtained whenever $x = -1$ or $x = 3$. To find the rest of the solutions to the inequality, we must check one number from each of the intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$. We check the signs at $x = -2$, $x = 0$, and $x = 4$.

When $x = -2$, we have

$$(-2)^2 - 2(-2) - 3 = (-2 - 3)(-2 + 1) = (-5)(-1) = 5 > 0.$$

This tells us that $x^2 - 2x - 3$ is positive on the interval $(-\infty, -11)$

When $x = 0$, we have

$$0^2 - 2(0) - 3 = -3 < 0.$$

This tells us that $x^2 - 2x - 3$ is negative on the interval $(-1, 3)$.

When $x = 4$, we have

$$(4)^2 - 2(4) - 3 = (4 - 3)(4 + 1) = (1)(5) = 5 > 0.$$

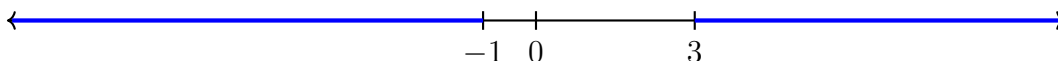This tells us that $x^2 - 2x - 3$ is positive on the interval $(3, \infty)$.

Combining this information, we have found that the inequality

$$0 \le x^2 - 2x - 3$$

is true on the intervals $(-\infty, -1)$ and $(3, \infty)$. In interval notation, the solutions are therefore

$$(-\infty, -1) \cup (3, \infty).$$

We can represent this union of intervals on the number line as

12 (Bonus - 10 Points)**.** Consider the function

$$p(x) = x^2 + 5x + 6.$$

(a) Find two numbers, $a < b$, for which the net change of p from a to b is 6.

Solution. This question simply asks that we find two values $a < b$ such that

$$f(b) - f(a) = 6.$$

The easiest possible solution would be to find a and b such that $f(b) = 6$ and $f(a) = 0$. This is equivalent to solving the two equations

$$x^2 + 5x + 6 = 6 \text{ and } x^2 + 5x + 6 = 0$$

for x and identifying a solution to the first equation which is larger than some solution to the second equation.

To solve the first, we subtract 6 from both sides to obtain

$$x^2 + 5x = x(x + 5) = 0$$

and note that this means either $x = 0$ or $x = -5$. For the second, we can factor

$$x^2 + 5x + 6 = (x + 3)(x + 2) = 0$$

to see that either $x = -3$ or $x = -2$.

By some luck, we find two possible solutions. If we take $a = -2$, then we can take $b = 0$ since

$$f(0) - f(-2) = 6 - 0 = 6.$$

Similarly, if we take $a = -3$, then we can also take $b = 0$ since

$$f(0) - f(-3) = 6 - 0 = 0.$$

(b) Find two numbers, $c < d$, for which the net change of p from c to d is -6.

[Hint: Your method from part (a) should also give you these values.]

Solution. Again, we observe that the easiest thing we could do is find two numbers, $c < d$, such that $f(d) = 0$ and $f(c) = 6$ so that the net change is

$$f(d) - f(c) = 0 - 6 = -6.$$

Using our work from part (a), we can take $c = -5$ and either $d = -2$ or $d = -3$ since

$$f(-2) - f(-5) = 0 - 6 = -6$$

and

$$f(-3) - f(-5) = 0 - 6 = -6.$$

Remark. There may well be many more solutions. These are just the easiest such solutions.