MATH 122
EXAM 01

BLAKE FARMAN
UNIVERSITY OF SOUTH CAROLINA

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period. If you require extra space, use the back of the page and indicate that you have done so.
Unless otherwise stated, all supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.** You may use a calculator **without a CAS** if you like, but a calculator is not necessary. By writing your name on the line below, you acknowledge that you have read and understand these directions.

Name: Solutions V1

Definitions	Points Earned	Points Possible	Problems	Points Earned	Points Possible
1		2	1		6
2		5	2		9
3		5	3		12
4		5	4		15
5		6	5		15
Subtotal		23	6		20
			Subtotal		77
			Total		100

Date: February 7, 2017.

1. Definitions

1 (2 Points). (a) State the Point-Slope form of a line passing through the point (x_0, y_0) with slope m.

$$y - y_0 = m(x - x_0)$$

(b) State the Slope-Intercept form of a line with slope m and y-intercept b.

$$y = mx + b$$

2 (5 Points). Let f be a function and let $a < b$ be given. State the average rate of change of f on the interval $[a, b]$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}.$$

3 (5 Points). Given a quantity P, state the relative change of the quantity from P to P'.

$$\frac{P'-P}{P}$$

4 (5 Points). (a) State the form of an exponential function of a variable t with inital value P_0 and base a:

$$P(t) = \underline{P_0 a^t}.$$

(b) The relative rate of change of P is

$$r = \underline{a-1}.$$

$$\frac{P(t+1)-P(t)}{P(t)} = \frac{P_0a^{t+1} - P_0a^t}{P_0a^t} = \frac{P_0a^t(a-1)}{P_0a^t} = a-1$$

[Hint: If you don't recall the formula, this is just the relative change from $P(t)$ to $P(t+1)$.]

(c) The function P models

- exponential growth when r is <u>positive</u>,
- exponential decay when r is <u>negative</u>.

(d) The continuous growth/decay rate is

$$k = \underline{\ln(a)}.$$

5 (6 Points). Let $0 < x, 0 < y$ be given. Fill in the blanks:

(i) $\ln(1) = \underline{0}$,

(ii) $\ln(xy) = \underline{\ln(x) + \ln(y)}$,

(iii) $\ln\left(\frac{x}{y}\right) = \underline{\ln(x) - \ln(y)}$,

(iv) $\ln(x^r) = \underline{r\ln(x)}$,

(v) $\ln(e^x) = \underline{x}$,

(vi) $e^{\ln(x)} = \underline{x}$.

2. Problems

1 (6 Points). (a) Find the slope of the line passing through the points $\left(\frac{1}{2}, 3\right)$ and $(1, 2)$.

$$m = \frac{3-2}{\frac{1}{2}-1} = \frac{1}{\left(\frac{-1}{2}\right)} = 1\left(\frac{-2}{1}\right) = -2$$

(b) Write the equation of this line in Point-Slope Form.

$$y - 3 = (-2)\left(x - \tfrac{1}{2}\right)$$

or

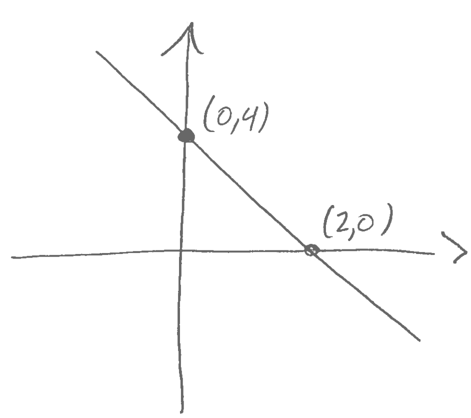
$$y - 2 = -2(x-1)$$

(c) Write the equation of this line in Slope-Intercept Form.

$$y - 3 = -2(x - \tfrac{1}{2})$$
$$\Rightarrow y = -2x + 1 + 3$$
$$= -2x + 4$$

$$y - 2 = -2(x-1)$$
$$\Rightarrow y = -2x + 2 + 2$$
$$= -2x + 4.$$

(d) Sketch a graph of $f(x)$. Label the x-intercept and the y-intercept.



2 (9 Points). Let $f(x) = x^2 - 1$.

(a) Compute the average rate of change for f between $x = 3$ and $x = 5$.

$$\frac{f(5)-f(3)}{5-3} = \frac{(5^2-1)-(3^2-1)}{2} = \frac{24-8}{2} = \frac{16}{2} = 8$$

(b) Give the Point-Slope form of the line that passes through $(3, f(3))$ and $(5, f(5))$.

$y-24=8(x-5)$ or $y-8=8(x-3)$

(c) Give the Slope-Intercept form of the line that passes through $(3, f(3))$ and $(5, f(5))$.

$y-24=8(x-5)$

$\Rightarrow y = 8x-40+24$

$= 8x-16$

$y-8=8(x-3)$

$\Rightarrow y = 8x-24+8$

$= 8x-16.$

3 (12 Points). A biologist observes a population with initial size 81. In two years, the biologist returns to observe the population again and finds that only 9 remain.

(a) Find an exponential function for the size of the population as a function of t years since the initial observation.

$$P(t) = 81a^t$$

$$P(2) = 81a^2 = 9$$

$$\Rightarrow a^2 = \frac{9}{81} = \frac{1}{9}$$

$$\Rightarrow a = \frac{1}{3}$$

So $P(t) = 81\left(\frac{1}{3}\right)^t$.

(b) Does the function from part (a) model growth or decay?

Decay

(c) Use the model from part (a) to determine how many years it will take for the size of the population to reach 1.

Solve $P(t) = 81\left(\frac{1}{3}\right)^t = 1$ for t.

$$81\left(\tfrac{1}{3}\right)^t = 1$$

$$\Rightarrow \left(\tfrac{1}{3}\right)^t = \frac{1}{81} = \frac{1}{9^2} = \frac{1}{(3^2)^2} = \frac{1}{3^4} = \left(\tfrac{1}{3}\right)^4$$

$$\Rightarrow \ln\left(\left(\tfrac{1}{3}\right)^t\right) = \ln\left(\left(\tfrac{1}{3}\right)^4\right)$$

$$t\ln\left(\tfrac{1}{3}\right) \qquad 4\ln\left(\tfrac{1}{3}\right)$$

$$\Rightarrow t = \frac{4\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{3}\right)} = 4.$$

4 (15 Points). A bank is offering an account that pays 5% interest compounded continuously. If you decide to invest money in this account, how long will it take for your initial investment to double? Give an **exact** answer.

The balance of the account after t years is

$$P(t) = P_0 e^{5/100 t} = P_0 e^{\frac{1}{20}t} = P_0 e^{t/20}.$$

Let d be the doubling time, so

$$P(d) = 2P_0 = P_0 e^{d/20}$$

$$\Rightarrow \quad 2 = e^{d/20}$$

$$\Rightarrow \quad \ln(2) = \ln\left(e^{d/20}\right) = d/20$$

$$\Rightarrow \quad \boxed{20\ln(2) = d.}$$

5 (15 Points). Sketch a graph of the function

$$f(x) = -2x^2 + 8x - 6.$$

Label any x-intercepts, the y-intercept, and the vertex.

$$f(x) = -2(x^2 - 4x + 3)$$
$$= -2(x-3)(x-1)$$

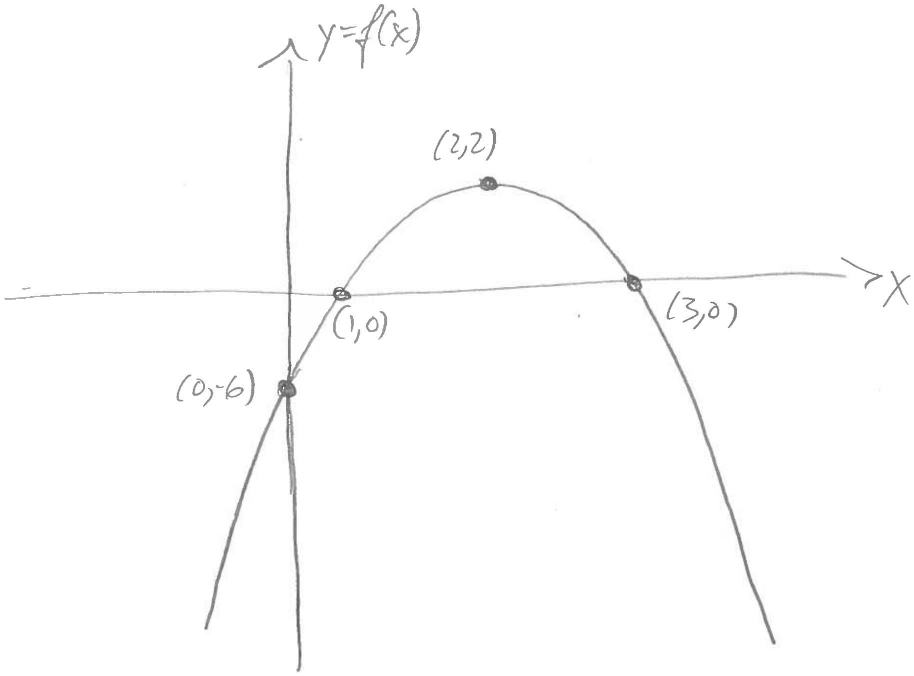
so the x-intercepts are $x=3$ and $x=1$. The vertex has x-coordinate

$$\frac{-8}{2(-2)} = \frac{8}{4} = 2$$

and y-coordinate

$$f(2) = -2(2-3)(2-1)$$
$$= -2(-1)(1)$$
$$= 2.$$

The y-intercept is $f(0) = -2(0)^2 + 8(0) - 6 = -6$.



6 (20 Points). A company hosts a weekly event. They find that 30 people attend at a ticket price of \$25, and 50 people attend at a ticket price of \$15. Assuming this relationship is linear, determine the ticket price that will generate the highest revenue.

The quantity, Q, of people attending the event as a function of the price, p, is the line through (25,30) with slope

$$m = \frac{30-50}{25-15} = \frac{-20}{10} = -2,$$

$$Q - 30 = -2(p-25)$$

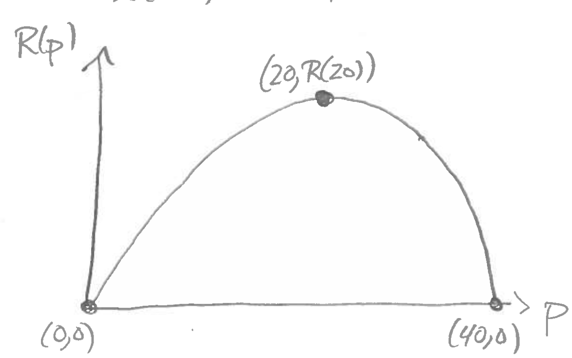
$$\Rightarrow Q(p) = -2p + 50 + 30 = -2p + 80.$$

The revenue as a function of price is therefore

$$R(p) = p \cdot Q(p) = p(-2p+80) = -2p^2 + 80p.$$

This is a downward facing parabola with p-intercepts $p=0$, $p=40$, and vertex at

$$p = \frac{-80}{2(-2)} = \frac{80}{4} = 20$$



Therefore the ticket price $p = \$20$ maximizes revenue.