MATH 142: EXAM 01

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period. Unless otherwise stated, all supporting work is required. It is advised, although not required, that you check your answers. You may **not** use any calculators.

Name: Answer Key

Problem	Points Earned	Points Possible
1		20
2		20
3		20
4		20
5		20
Bonus		10
Total		100

Date: July 11, 2014.

1. Problems

For each of the following problems, decide which method of integration is appropriate and compute the given integrals. You will find some useful trigonmetric identities on the last page. If you need more space for a problem, use the back of the page.

1 (20 Points). *Compute the following integrals.*

(a) $\int \theta \cos(\theta^2) d\theta.$

$u = \theta^2$

$du = 2\theta d\theta$

$\Rightarrow \frac{1}{2} du = \theta d\theta$

$$\int \theta \cos(\theta^2) d\theta = \frac{1}{2} \int \cos(u)\, du$$

$$= \boxed{\frac{1}{2} \sin(\theta^2) + C}$$

(b) $\int \theta^3 \cos(\theta^2) d\theta.$

$$\int \theta^3 \cos(\theta^2)\, d\theta = \int \theta^2\, \theta \cos(\theta^2)\, d\theta$$

$u = \theta^2$ $\quad$ $v = \frac{1}{2}\sin(\theta^2)$

$du = 2\theta d\theta$ $\quad$ $dv = \theta \cos(\theta^2) d\theta$

$$= \frac{1}{2}\theta^2 \sin(\theta^2) - \frac{1}{2}\int 2\theta \sin(\theta^2) d\theta$$

$u = \theta^2$

$du = 2\theta d\theta$

$$= \frac{1}{2}\theta^2 \sin(\theta^2) - \frac{1}{2}\int \sin(u)\, du$$

$$= \boxed{\frac{1}{2}\theta^2 \sin(\theta^2) + \frac{1}{2}\cos(\theta^2) + C}$$

2 (20 Points). *Compute* $\int \frac{dx}{\sqrt{x^2+16}}$.

Let $x = 4\tan(\theta)$, so

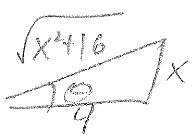
$$dx = 4\sec^2(\theta)d\theta$$

$$\sqrt{x^2+16} = \sqrt{16\tan^2(\theta)+16} = \sqrt{16(\tan^2(\theta)+1)} = \sqrt{16\sec^2(\theta)} = 4\sec(\theta)$$

So

$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2(\theta)d\theta}{4\sec(\theta)} = \int \sec(\theta)d\theta = \ln|\sec(\theta)+\tan(\theta)|+C.$$

From $\frac{x}{4} = \tan(\theta)$, we have the triangle



which gives

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\sqrt{x^2+16}}{4}.$$

Therefore, substituting for $\sec(\theta)$ and $\tan(\theta)$, we get

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C.$$

3 (20 Points). *Compute* $\int \cos^2(\theta)\tan^3(\theta)d\theta$.

Write

$$\cos^2(\theta)\tan^3(\theta) = \cos^2(\theta)\frac{\sin^3(\theta)}{\cos^3(\theta)} = \frac{\sin^3(\theta)}{\cos(\theta)}$$

so

$$\int \cos^2(\theta)\tan^3(\theta)d\theta = \int \frac{\sin^3(\theta)}{\cos(\theta)}d\theta$$

$$= \int \frac{\sin^2(\theta)\sin(\theta)d\theta}{\cos(\theta)}$$

$$= \int \frac{(1-\cos^2(\theta))\sin(\theta)\,d\theta}{\cos(\theta)}$$

$u = \cos(\theta)$

$du = -\sin(\theta)d\theta$

$\Rightarrow -du = \sin(\theta)d\theta$

$$= \int \frac{1-u^2}{u}(-du)$$

$$= \int \frac{u^2-1}{u}du$$

$$= \int u\,du - \int \frac{1}{u}du$$

$$= \frac{1}{2}u^2 - \ln|u| + C$$

$$= \boxed{\frac{1}{2}\cos^2(\theta) - \ln|\cos(\theta)| + C}$$

4 (20 Points). *Compute* $\int 2x\tan(x^2)dx$.

$u = x^2$
$du = 2x\,dx$

$$\int 2x\tan(x^2)\,dx = \int \tan(u)\,du$$

$$= \int \frac{\tan(u)\sec(u)}{\sec(u)}\,du$$

$w = \sec(u)$
$dw = \sec(u)\tan(u)\,du$

$$= \int \frac{dw}{w}$$

$$= \ln|w| + C$$

$$= \ln|\sec(u)| + C$$

$$= \boxed{\ln|\sec(x^2)| + C.}$$

5 (20 Points). *Compute* $\int \frac{x}{x^2+x-2}dx$.

Observe that we may factor

$$x^2+x-2 = (x-1)(x+2)$$

so using the method of partial fractions

$$\frac{x}{x^2+x-2} = \frac{x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}.$$

Finding a common denominator on the right we have

$$\frac{x}{x^2+x-2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)} = \frac{x(A+B)+(2A-B)}{(x-1)(x+2)}.$$

Clearing denominators gives

$$x = x(A+B)+2A-B$$

and comparing coefficients we have

$$A+B=1$$
$$2A-B=0.$$

Then $A=\frac{1}{2}B$, so $1=\frac{1}{2}B+B=\frac{3}{2}B$ gives $B=2/3$; and $A=\frac{1}{2}\left(\frac{2}{3}\right)=\frac{1}{3}$.
Therefore

$$\int\frac{x}{x^2+x-2}dx = \int\left(\frac{A}{x-1}+\frac{B}{x+2}\right)dx$$
$$= A\int\frac{dx}{x-1} + B\int\frac{dx}{x+2}$$
$$= \frac{1}{3}\int\frac{dx}{x-1} + \frac{2}{3}\int\frac{dx}{x+2}$$
$$= \boxed{\frac{1}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + C}$$

6 (Bonus - 10 points). *Compute* $\int \sqrt{\frac{1-x}{1+x}}dx$.

First observe that $\sqrt{\frac{1-x}{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1-x}}{\sqrt{1-x}} = \frac{1-x}{\sqrt{(1+x)(1-x)}} = \frac{1-x}{\sqrt{1-x^2}}$.

Then

$$\int\sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx.$$

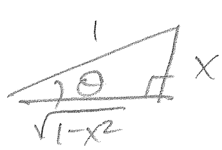
Let $x = \sin(\theta)$, so $dx = \cos(\theta)d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta).$$

Then

$$\begin{aligned}\int\sqrt{\frac{1-x}{1+x}}\,dx &= \int \frac{(1-\sin(\theta))}{\cos(\theta)}\cos(\theta)d\theta \\ &= \int (1-\sin(\theta))\,d\theta \\ &= \int d\theta - \int \sin(\theta)d\theta \\ &= \theta + \cos(\theta) + C.\end{aligned}$$

We use the triangle



to see that $\cos(\theta) = \sqrt{1-x^2}$ and

$$\theta = \arcsin(\sin(\theta)) = \arcsin(x).$$

Therefore

$$\int\sqrt{\frac{1-x}{1+x}}\,dx = \arcsin(x) + \sqrt{1-x^2} + C.$$