MATH 142: EXAM 03

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the class period. Unless otherwise stated, all supporting work is required. You may **not** use any calculators.

Name: Answer Key

Problem	Points Earned	Points Possible
1		20
2		20
3		20
4		20
5		20
Bonus		10
Total		100

Date: July 25, 2014.

1. Problems

1. *Find the area between the curves* $y = \sin\left(\frac{\pi x}{2}\right)$ *and* $y = x$.

These curves intersect at $(-1,-1)$, $(0,0)$, and $(1,1)$.

Since $\sin\left(\frac{-\pi x}{2}\right) - (-x) = -\sin\left(\frac{\pi x}{2}\right) + x = -\left(\sin\left(\frac{\pi x}{2}\right) - x\right)$

we see that $\sin\left(\frac{\pi x}{2}\right) - x$ is an odd function. This tells us that

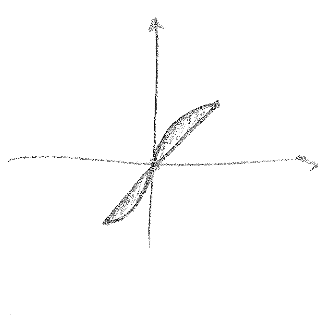
$$0 = \int_{-1}^{1}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx = \int_{-1}^{0}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx + \int_{0}^{1}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx$$

so

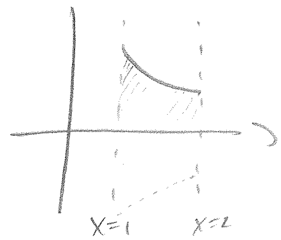
$$\int_{-1}^{0}\left(x - \sin\left(\frac{\pi x}{2}\right)\right)dx = -\int_{-1}^{0}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx = \int_{0}^{1}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx.$$

Since $\sin\left(\frac{\pi x}{2}\right) > x$ on $[0,1]$ and $x > \sin\left(\frac{\pi x}{2}\right)$ on $[-1,0]$, the area is

$$\begin{aligned}
\int_{-1}^{0}\left(x - \sin\left(\frac{\pi x}{2}\right)\right)dx + \int_{0}^{1}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx &= 2\int_{0}^{1}\left(\sin\left(\frac{\pi x}{2}\right) - x\right)dx \\
&= 2\left[\frac{-2}{\pi}\cos\left(\frac{\pi x}{2}\right)\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1\right] \\
&= -\frac{4}{\pi}(0-1) - (1-0) \\
&= \frac{4}{\pi} - 1 \\
&= \boxed{\frac{4-\pi}{\pi}.}
\end{aligned}$$



2. *Find the volume of the solid obtained by rotating the region bounded by the curves* $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, *and* $y = 0$ *about the* x*-axis.*



The area of a disc at x is $A(x) = \pi\left(\frac{1}{x}\right)^2$

so the volume is

$$\int_1^2 A(x)\,dx = \pi\int_1^2 x^{-2}\,dx$$

$$= \pi\,(-1)\,x^{-1}\Big|_1^2$$

$$= -\pi\left(\tfrac{1}{2} - 1\right)$$

$$= \boxed{\pi/2}.$$

3. *Find a power series representation for* $\frac{1}{x-5}$. *Once you have found this power series, find its radius of convergence and the interval of convergence.*

$$\frac{1}{x-5} = -\frac{1}{5}\frac{1}{(1-\frac{x}{5})}$$
$$= -\frac{1}{5}\sum_{n=0}^{\infty}(x/5)^n$$
$$= -\sum_{n=0}^{\infty} x^n/5^{n+1}.$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+2}}\cdot\frac{5^{n+1}}{x^n}\right| = \frac{|x|}{5} < 1 \iff |x| < 5 = R.$$

When $x=-5$,

$$\sum_{n=0}^{\infty}\frac{(-5)^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{5}$$

diverges because $\lim_{n\to\infty}(-1)^n/5$ does not exist

When $x=5$,

$$\sum_{n=0}^{\infty} 5^n/5^{n+1} = \sum_{n=0}^{\infty} 1/5$$

diverges because $\lim_{n\to\infty} 1/5 = 1/5 \neq 0$.

Therefore the interval of convergence is $(-5,5)$.

4. *(a) Find the Maclaurin expansion (i.e. find the Taylor expansion about $a = 0$) for the function $f(x) = \cos(x)$. Find the radius of convergence and the interval of convergence for this series.*

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f'''(0) = \sin(0) = 0$

$f^{(iv)}(0) = \cos(0) = 1$

$\vdots$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \frac{(2n)!}{x^{2n}} \right| = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0,$$

so $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

(b) Use Taylor's Inequality to show that $\cos(x)$ is equal to its Maclaurin expansion on the interval of convergence you found in Part a. You will find the statement of Taylor's Inequality on the last page.

Observe that $|f^{(n+1)}(x)| = \begin{cases} |\pm\cos(x)| & n \text{ even} \\ 0 & \text{else} \end{cases} \leq 1$

so by Taylor's Inequality

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{1}{(n+1)!} |x|^{n+1} = 0,$$

so $|R_n(x)| \to 0$ as $n \to \infty$ (Squeeze Theorem)

and thus $R_n(x) \to 0$ as $n \to \infty$. $\blacksquare$

5. *Find the Maclaurin expansion of the function* $f(x) = xe^x$.

$$f^{(n)}(x) = (n+x)e^x$$

$$f^{(n)}(0) = n$$

$$xe^x = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}$$