CALCULUS OF INVERSE FUNCTIONS

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Name: Solutions

1. Find the limit: $\lim_{x\to\infty} 3^x$ $= \infty$

2. Find the limit: $\lim_{x\to-\infty} \left(\frac{1}{3}\right)^x$ $= \lim_{x\to\infty} \left(\frac{1}{3}\right)^{-x} = \lim_{x\to\infty} 3^x = \infty$

3. Find $\frac{dy}{dx}$. Assume y is a differentiable function of x.

$$3y = xe^{5y}$$

$$\ln(3y) = \ln(3) + \ln(y) = \ln(xe^{5y}) = \ln(x) + \ln(e^{5y}) = \ln(x) + 5y$$

$$\Rightarrow \ln(y) - 5y = \ln(x) - \ln(3)$$

$$\Rightarrow \frac{y'}{y} - 5y' = \frac{1}{x} \Rightarrow y' - 5yy' = \frac{y}{x} \Rightarrow y'(1-5y) = \frac{y}{x}$$

$$\Rightarrow y' = \boxed{\frac{y}{x(1-5y)}}$$

For problems 4-10, find $f'(x)$.

4. $f(x) = e^x \sin x$

$$f'(x) = \boxed{e^x \sin(x) + e^x \cos(x)}$$

5. $f(x) = \ln(xe^{7x}) = \ln(x) + \ln(e^{7x}) = \ln(x) + 7x$

$$f'(x) = \boxed{\frac{1}{x} + 7}$$

6. $f(x) = \dfrac{x}{\sqrt{1 - \ln(x)^2}}$ $\qquad \ln(f(x)) = \ln\left(\frac{x}{\sqrt{1-\ln(x)^2}}\right) = \ln(x) - \frac{1}{2}\ln\left(1-\ln(x)^2\right)$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{1}{2}\left(\frac{-2\ln(x)\left(\frac{1}{x}\right)}{1-\ln(x)^2}\right)$$

$$\Rightarrow f'(x) = f(x)\left(\frac{1}{x} + \frac{\ln(x)}{x(1-\ln(x)^2)}\right) = \boxed{\frac{x}{\sqrt{1-\ln(x)^2}}\left(\frac{1}{x} + \frac{\ln(x)}{x(1-\ln(x)^2)}\right)}$$

7. $f(x) = (xe^x)^\pi = x^\pi e^{\pi x}$

$$f'(x) = \pi x^{\pi-1} e^{\pi x} + x^\pi (\pi e^{\pi x})$$
$$= \boxed{\pi\left(x^{\pi-1} + x^\pi\right) e^{\pi x}}$$

8. $f(x) = (e^{2x} + e)^{\frac{1}{2}}$

$$f'(x) = \boxed{\frac{2e^{2x}}{2\sqrt{e^{2x}+e}}}$$

9. $f(x) = \big(\ln(5x^2 + 9)\big)^3$

$$f'(x) = 3\ln(5x^2+9)^2\left(\frac{10x}{5x^2+9}\right)$$
$$= \boxed{\frac{30x\ln(5x^2+9)^2}{5x^2+9}}$$

10. $f(x) = \ln\left((5x^2+9)^3\right)$ $= 3\ln(5x^2+9)$

$$f'(x) = \frac{3(10x)}{5x^2+9} = \boxed{\frac{30x}{5x^2+9}}$$

For problems 11-20, find the indefinite integral (you may need u-substitution).

11. $\int e^x \, \mathrm{d}x$ $= e^x + C$

12. $\int a^x \, \mathrm{d}x$, where $a > 0$ is a constant ($\neq 1$).

$$\int a^x dx = \boxed{\frac{a^x}{\ln(a)} + C}$$

13. $\int \pi^{2x} \, \mathrm{d}x$ $= \frac{1}{2}\int \pi^u du = \frac{1}{2} \frac{\pi^u}{\ln(\pi)} + C$

$u = 2x$

$\frac{1}{2}du = \frac{1}{2}(2dx) = dx$

$$= \boxed{\frac{\pi^{2x}}{2\ln(\pi)} + C}$$

14. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$

15. $\int e^{2x}\,\mathrm{d}x = \frac{e^{2x}}{2\ln(e)} + C = \boxed{\frac{1}{2}e^{2x} + C}$

16. $\int \frac{\ln x}{x}\,\mathrm{d}x = \int u\,du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\ln(x)^2 + C}$

$u = \ln(x)$

$du = \frac{1}{x}dx$

17. $\int \frac{\sqrt{\ln(x)}}{x}\,\mathrm{d}x = \int u^{1/2} du = \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{3}\ln(x)^{3/2} + C}$

$u = \ln(x)$

$du = \frac{1}{x}dx$

18. $\int \frac{e^x}{\sqrt{1-e^x}}\,\mathrm{d}x = -\int \frac{1}{\sqrt{u}} du = -2\sqrt{u} + C = \boxed{-2\sqrt{1-e^x} + C}$

$u = 1-e^x$

$-du = -(-e^x dx)$

$= e^x dx$

19. $\int \frac{\ln(e^{2x})}{x^2}\,\mathrm{d}x = \int \frac{2x}{x^2} dx = 2\int \frac{1}{x} dx = \boxed{2\ln|x| + C}$

20. $\int \frac{e^x}{3+e^x}\,\mathrm{d}x$ $= \int \frac{1}{u}du = \ln|u|+C$

$u = 3+e^x$

$du = e^x dx$

$= \ln|3+e^x|+C$

$= \boxed{\ln(3+e^x)+C}$

21. Evaluate the definite integral: $\int_2^3 \frac{xe^{x^2}}{3}\,\mathrm{d}x$

$u = x^2$ $\quad u(2)=4,\ u(3)=9$

$du = 2x\,dx$

$\Rightarrow \frac{1}{6}du = \frac{x}{3}dx$

$\int_2^3 \frac{xe^{x^2}}{3}dx = \frac{1}{6}\int_4^9 e^u du = \boxed{\frac{1}{6}(e^9-e^4)}$