CHAIN RULE

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Name: Solutions

Theorem (The Chain Rule)**.** *Assume that g is differentiable at x and f is differentiable at $g(x)$. Then composition of f with g, $f \circ g(x) = f\left(g(x)\right)$, is differentiable at x and*

$$\frac{\mathrm{d}}{\mathrm{d}x} f \circ g(x) = f'\left(g(x)\right) \cdot g'(x).$$

1. Let $f(x) = (3x^2+1)^2$.

(a) Expand $f(x)$, then take the derivative.

$$f(x) = (3x^2)^2 + 2(3x^2)(1) + 1^2 = 9x^4 + 6x^2 + 1$$

$$f'(x) = 36x^3 + 12x$$

(b) Write $f(x) = (3x^2+1)^2 = (3x^2+1)(3x^2+1)$ and apply the Product Rule.

$$\begin{aligned} f'(x) &= 6x(3x^2+1) + (3x^2+1)6x \\ &= 2(18x^3+6x) \\ &= 36x^3 + 12x \end{aligned}$$

(c) Apply the chain rule directly to $f(x)$.

$$f'(x) = 2(3x^2+1)(6x) = 2(18x^3+6x) = 36x^3 + 12x$$

(d) Are your answers in parts (a), (b), and (c) the same? Why or why not?

They are the same because each are valid ways to compute the derivative.

2. Assume that f is a differentiable function and let $g(x) = \left(f\left(\sqrt{x}\right)\right)^3$.

(a) Compute $g'(x)$. Your answer should include both f and f'.

$$g'(x) = 3\left(f(\sqrt{x})\right)^2 \frac{d}{dx} f(\sqrt{x})$$
$$= 3\left(f(\sqrt{x})\right)^2 f'(\sqrt{x}) \frac{d}{dx}\sqrt{x}$$
$$= 3\left(f(\sqrt{x})\right)^2 f'(\sqrt{x}) \left(\frac{1}{2\sqrt{x}}\right)$$

(b) If $f(2) = 1$ and $f'(2) = -2$, calculate $g'(4)$.

$$g'(4) = 3\left(f(\sqrt{4})\right)^2 f'(\sqrt{4}) \frac{1}{2\sqrt{4}} = 3\left(f(2)\right)^2 f'(2) \frac{1}{4}$$
$$= \frac{3(1)^2(-2)}{4} = \frac{-6}{4} = \boxed{\frac{-3}{2}}$$

Find the derivative of the given function.

3. $f(x) = (1 + x + x^2)^{99}$

$$f'(x) = \boxed{99(1+x+x^2)^{98}(1+2x)}$$

4. $g(\theta) = (2 - \sin(\theta))^{3/2}$

$$g'(\theta) = \frac{3}{2}(2-\sin(\theta))^{1/2}(-\cos(\theta))$$
$$= \boxed{\frac{-3\cos(\theta)\sqrt{2-\sin(\theta)}}{2}}$$

5. $g(\theta) = \cos^2(\theta)$

$$g'(\theta) = 2\cos(\theta)(-\sin(\theta))$$

$$= \boxed{-2\sin(\theta)\cos(\theta)}$$

6. $f(\theta) = \cot^2(\sin(\theta))$

$$f'(\theta) = 2\cot(\sin(\theta))\left(-\csc^2(\sin(\theta))\cos(\theta)\right)$$

$$= \boxed{-2\cot(\sin(\theta))\csc^2(\sin(\theta))\cos(\theta)}$$