INTEGRATION

BLAKE FARMAN

Lafayette College

Name: Solutions

Use right endpoints and the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

to evaluate the following integrals.

1.

$$\int_0^3 x^3 \, \mathrm{d}x.$$

$f(x) = x^3$

$\Delta x = \frac{3-0}{n} = \frac{3}{n}, \quad x_i = a + i\Delta x = 0 + i\,3/n = \frac{3i}{n}$

$f\left(\frac{3i}{n}\right)\Delta x = \frac{27i^3}{n^3}\left(\frac{3}{n}\right) = \frac{81i^3}{n^4}$

$$\int_0^3 x^3 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{81i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{81}{n^4} \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{81}{4} \frac{n^2+2n+1}{n^2} = \lim_{n\to\infty} \frac{81}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{81}{4}(1+0+0)$$

$$= \boxed{\frac{81}{4}}$$

2.

$$\int_1^2 x^2 \, \mathrm{d}x$$

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 1 + i/n$$

$$f(x_i)\Delta x = (1+i/n)^2(1/n) = (1 + 2i/n + i^2/n^2)\,1/n$$

$$= \frac{1}{n} + \frac{2}{n}i + \frac{1}{n^3}i^2$$

$$\int_1^2 x^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{2}{n^2}i + \frac{1}{n^3}i^2\right)$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} \frac{2}{n^2}i + \sum_{i=1}^{n} \frac{1}{n^3}i^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}(n) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2} \right]$$

$$= \lim_{n\to\infty} \left[1 + 1 + \frac{1}{n} + \frac{2n^2+3n+1}{6n^2} \right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{1}{n} + \frac{2n^2}{6n^2} + \frac{3n}{6n^2} + \frac{1}{6n^2} \right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{1}{n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right]$$

$$= 2 + 0 + \frac{1}{3} + 0 + 0$$

$$= \boxed{\frac{7}{3}}$$