DERIVATIVES AND SHAPE

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Name: Solutions

Critical Numbers

Find the critical numbers of the given function

1. $f(x) = x^3 + 6x^2 - 15x$

$$f'(x) = 3x^2 + 12x - 15$$
$$= 3(x^2 + 4x - 5)$$
$$= 3(x+5)(x-1)$$
$$= 0$$
$$\Rightarrow \boxed{x = 1 \text{ or } x = -5}$$

2. $f(x) = 2x^3 + x^2 + 2x$

$$f'(x) = 6x^2 + 2x + 2$$
$$= 2(3x^2 + x + 1)$$

Since $(1)^2 - 4(3)(1) = 1 - 12 = -11 < 0$ there are no critical points.

3. $f(x) = |3x - 4|$

$$f(x) = \begin{cases} 3x-4 & \text{if } 0 \le 3x-4 \\ -(3x-4) & \text{if } 3x-4 < 0 \end{cases} = \begin{cases} 3x-4 & \text{if } \frac{4}{3} \le x \\ -3x+4 & \text{if } x < 4/3 \end{cases}$$

Because $|x|$ is not differentiable at $x=0$, f is not differentiable when $3x-4=0$, or $x = 4/3$. So

$$f'(x) = \begin{cases} 3 & \text{if } \frac{4}{3} < x \\ -3 & \text{if } x < 4/3 \\ \text{undef.} & \text{if } x = 4/3 \end{cases}$$

implies that $\boxed{x = 4/3}$ is the only critical point.

4. $f(x) = \dfrac{x-1}{x^2-x+1}$

$$f'(x) = \frac{x^2-x+1-(x-1)(2x-1)}{(x^2-x+1)^2} = \frac{x^2-x+1-2x^2+3x-1}{(x^2-x+1)^2}$$

$$= \frac{-x^2+2x}{(x^2-x+1)^2} = 0 \iff 0 = -x^2+2x = -x(x-2)$$

$$\iff \boxed{x=0 \quad \text{or} \quad x=2}$$

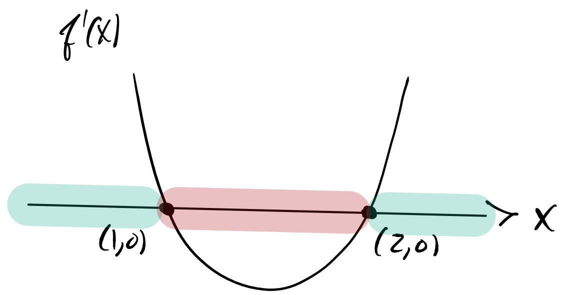
First Derivatives

For each function use the first derivative to find

- the interval(s) where the given function is increasing
- the interval(s) where the given function is decreasing
- the local maximum/minimum values

5. $f(x) = 2x^3 - 9x^2 + 12x - 3$

$$f'(x) = 6x^2 - 18x + 12$$
$$= 6(x^2 - 3x + 2)$$
$$= 6(x-2)(x-1)$$



Increasing : $(-\infty, 1) \cup (2, \infty)$

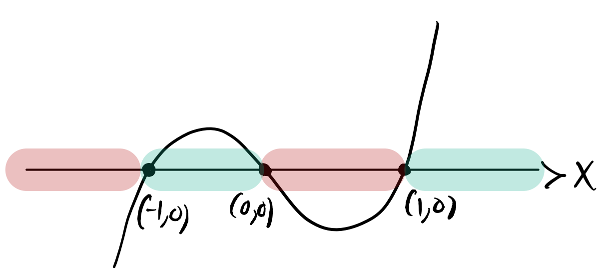
Decreasing : $(1, 2)$

Local maximum value : $f(1) = 2 - 9 + 12 - 3$
$= 2 + 3 - 3$
$= 2$

Local minimum value : $f(2) = 2(8) - 9(4) + 12(2) - 3$
$= 16 - 36 + 24 - 3$
$= 16 - 12 - 3$
$= 1$

6. $f(x) = x^4 - 2x^2 + 3$

$$f'(x) = 4x^3 - 4x$$
$$= 4x(x^2 - 1)$$
$$= 4x(x+1)(x-1)$$



Increasing : $(-1, 0) \cup (1, \infty)$

Decreasing : $(-\infty, -1) \cup (0, 1)$

Local maximum value : $f(0) = 3$

Local minimum values : $f(-1) = 1 - 2 + 3 = 2$

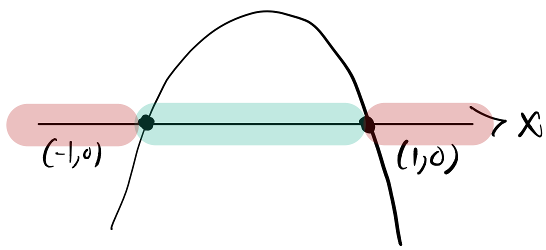
$f(1) = 1 - 2 + 3 = 2$

7. $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{x^2+1 - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

Since $(x^2+1)^2 > 0$, the sign of $f'(x)$ only depends on $-x^2+1 = -(x^2-1) = -(x+1)(x-1)$

$y = -x^2+1$



Increasing : $(-1,1)$

Decreasing : $(-\infty,1) \cup (1,\infty)$

Local maximum value : $f(1) = \frac{1}{2}$

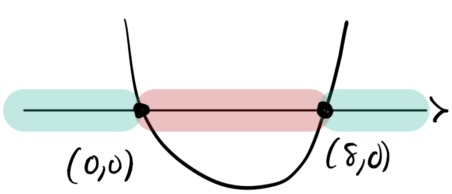
Local minimum value : $f(-1) = \frac{-1}{2}$

8. $f(x) = \dfrac{x^2}{x-4}$

$$f'(x) = \frac{2x(x-4) - x^2}{(x-4)^2} = \frac{x^2-8x}{(x-4)^2}$$

Since $(x-4)^2 \geq 0$, the sign of $f'(x)$ depends only on

$$x^2 - 8x = x(x-8)$$



Increasing : $(-\infty,0) \cup (8,\infty)$

Decreasing : $(0,8)$

Local maximum value : $f(0) = 0$

Local minimum value : $f(8) = \frac{64}{4} = 16$

Second Derivatives

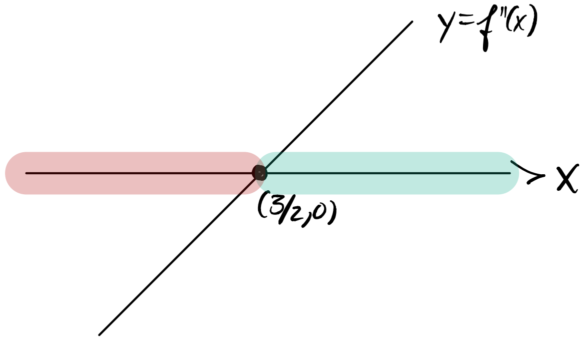
For each function use the second derivative to find

- the interval(s) where the given function is concave up
- the interval(s) where the given function is concave down
- the inflection points

9. $f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 6x^2 - 18x + 12$

$f''(x) = 12x - 18 = 6(2x-3)$



Concave Up: $\left(\frac{3}{2}, \infty\right)$

Concave down: $\left(-\infty, \frac{3}{2}\right)$

Inflection Point(s):

$f(3/2) = 2\left(\frac{27}{8}\right) - 9\left(\frac{9}{4}\right) + 12\left(\frac{3}{2}\right) - 3$

$= \frac{27}{4} - \frac{81}{4} + 18 - 3$

$= \frac{-54}{4} + \frac{60}{4} = \frac{6}{4} = \frac{3}{2}$

$\left(\frac{3}{2}, \frac{3}{2}\right)$

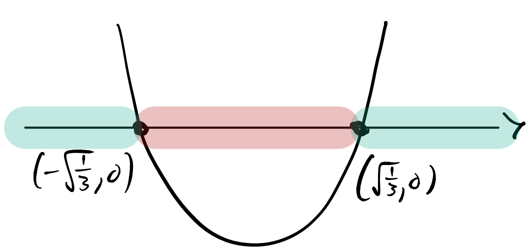
10. $f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4 = 0$

$\Rightarrow 3x^2 - 1 = 0$

$\Rightarrow x = \pm\sqrt{\frac{1}{3}}$



Concave Up: $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, 0\right)$

Concave down: $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$

Inflection Point(s):

Observe that f is even, so

$f\left(-\sqrt{\frac{1}{3}}\right) = f\left(\sqrt{\frac{1}{3}}\right) = \frac{1}{3^2} - \frac{2}{3} + 3 = \frac{1}{9} - \frac{6}{9} + \frac{27}{9}$

$= \frac{22}{9}$

$\left(-\sqrt{\frac{1}{3}}, \frac{22}{9}\right)$, $\left(\sqrt{\frac{1}{3}}, \frac{22}{9}\right)$

11. $f(x) = \dfrac{x}{x^2+1}$

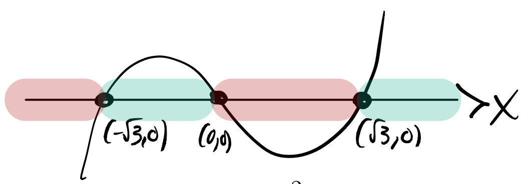
$$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x(x^2+1)^2 - (-x^2+1)(4x)(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1)(x^2+1-2x^2+2)}{(x^2+1)^4}$$

$$= \frac{-2x(-x^2+3)}{(x^2+1)^3}$$

Since $(x^2+1)^3 > 0$, the sign of $f''(x)$ depends only on $2x(x^2-3)$



Concave up: $(-\sqrt{3},0) \cup (\sqrt{3},\infty)$

Concave down: $(-\infty,-\sqrt{3}) \cup (0,\sqrt{3})$

Inflection Point(s):

$f(\sqrt{3}) = \dfrac{\sqrt{3}}{3+1} = \dfrac{\sqrt{3}}{4}$, $f(-\sqrt{3}) = \dfrac{-\sqrt{3}}{4}$

$(0,0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

12. $f(x) = \dfrac{x^2}{x-4}$ $\qquad f'(x) = \dfrac{x^2-8x}{(x-4)^2}$

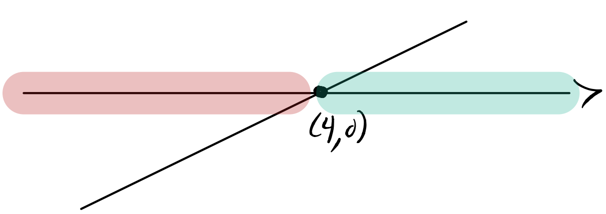
$$f''(x) = \frac{(2x-8)(x-4)^2 - (x^2-8x)2(x-4)}{(x-4)^4}$$

$$= \frac{2(x-4)((x-4)^2 - x^2 + 8x)}{(x-4)^4}$$

$$= \frac{2(x^2-8x+16-x^2+8x)}{(x-4)^3}$$

$$= \frac{32}{(x-4)^3}$$

The sign of $f''(x)$ depends only on $x-4$



Concave up: $(4,\infty)$

Concave down: $(-\infty,4)$

Inflection Point(s):

None!

There is no inflection point when $x=4$ because $f(x)$ has a vertical asymptote at $x=4$.