SUBSTITUTION

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Name: Solutions

Use substitution and the Fundamental Theorem of Calculus to evaluate the following integrals.

1. $\displaystyle\int_0^1 \cos(\pi t/2)\,\mathrm{d}t$

$u = \pi t/2$, $du = \pi/2\, dt \Rightarrow 2/\pi\, du = dt$

$u(0) = 0$, $u(1) = \pi/2$

$$\int_0^1 \cos(\pi t/2)\,dt = \int_0^{\pi/2} \cos(u)\, \tfrac{2}{\pi}\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$$= \frac{2}{\pi}\sin(u)\Big|_0^{\pi/2} = \frac{2}{\pi}\left(\sin\left(\tfrac{\pi}{2}\right) - \sin(0)\right)$$

$$= \frac{2}{\pi}(1-0) = \boxed{\frac{2}{\pi}}$$

2. $\displaystyle\int_0^1 (2t-1)^{50}\,\mathrm{d}t$

$u = 2t-1$, $du = 2dt \Rightarrow dt = \frac{1}{2}du$

$u(0) = -1$, $u(1) = 2(1)-1 = 1$

$$\int_0^1 (2t-1)^{50}\,dt = \int_{-1}^{1} u^{50}\left(\tfrac{1}{2}\right)du = \frac{1}{2}\int_{-1}^{1} u^{50}\,du = \frac{1}{2}\left(\frac{1}{51}\right)u^{51}\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{51}\right)\left(1^{51} - (-1)^{51}\right) = \frac{1}{2}\left(\frac{1}{51}\right)(1+1) = \frac{2}{2(51)}$$

$$= \boxed{\frac{1}{51}}$$

3. $\displaystyle\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}\,\mathrm{d}t$

$u = \cos(t)$, $du = -\sin(t)dt \Rightarrow -du = \sin(t)dt$

$u(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$, $u(0) = \cos(0) = 1$

$$\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}dt = \int_1^{\frac{\sqrt{3}}{2}} \frac{-du}{u^2} = -\int_1^{\frac{\sqrt{3}}{2}} u^{-2}du = -(-1)u^{-1}\Big|_1^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{u}\Big|_1^{\sqrt{3}/2} = \frac{1}{\frac{\sqrt{3}}{2}} - \frac{1}{1} = \boxed{\frac{2}{\sqrt{3}} - 1}.$$

4. $\displaystyle\int_0^{3} x\sqrt{9 - x^2}\,\mathrm{d}x$

$u = 9 - x^2$, $du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$u(0) = 9 - 0 = 9$, $u(3) = 9 - 9 = 0$

$$\int_0^3 x\sqrt{9-x^2}\,dx = \int_9^0 \sqrt{u}\left(\frac{-1}{2}\right)du$$

$$= \frac{-1}{2}\int_9^0 u^{1/2}du$$

$$= \frac{-1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_9^0$$

$$= \frac{-1}{3}\left(0^{3/2} - 9^{3/2}\right)$$

$$= \frac{-1}{3}\left(0 - (\sqrt{9})^3\right)$$

$$= \frac{-1}{3}(-3^3) = \frac{3^3}{3} = 3^2 = \boxed{9}$$

5. $\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,\mathrm{d}x$

$u = \sin(x)$, $du = \cos(x)du$

$u(0) = 0$, $u(\frac{\pi}{2}) = 1$

$$\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,dx = \int_0^1 \sin(u)\,du$$

$$= -\cos(u)\Big|_0^1$$

$$= -(\cos(1) - \cos(0))$$

$$= -(\cos(1) - 1)$$

$$= \boxed{1 - \cos(1)}$$

6. $\int_{-\pi/3}^{\pi/3} x^4 \sin(x)\,\mathrm{d}x = 0$ because $x^4\sin(x)$ is odd:

$f(x) = x^4\sin(x)$

$f(-x) = (-x)^4\sin(-x) = x^4(-\sin(x)) = -x^4\sin(x) = -f(x)$