REVIEW: FUNDAMENTAL THEOREM OF CALCULUS

BLAKE FARMAN

Lafayette College

Name: Solutions

Use the Fundamental Theorem of Calculus to evaluate the following definite integrals.

1. $\int_1^3 \left(x^2 + 2x - 4\right) \mathrm{d}x$

$$\int_1^3 (x^2+2x-4)dx = \int_1^3 x^2 dx + 2\int_1^3 x dx - 4\int_1^3 dx$$
$$= \frac{1}{3}x^3\Big|_1^3 + x^2\Big|_1^3 - 4x\Big|_1^3$$
$$= \frac{1}{3}(3^3-1) + (3^2-1) - 4(3-1)$$
$$= \frac{26}{3} + 8 - 8$$
$$= \boxed{\frac{26}{3}}$$

2. $\int_0^1 \left(1 - 8v^3 + 16v^7\right) \mathrm{d}v$

$$\int_0^1 (1-8v^3+16v^7)dx = \int_0^1 dv - 8\int_0^1 v^3 dv + 16\int_0^1 v^7 dv$$
$$= v\Big|_0^1 - 8\left(\frac{1}{4}\right)v^4\Big|_0^1 + 16\left(\frac{1}{8}\right)v^8\Big|_0^1$$
$$= (1-0) - 2(1-0) + 2(1-0)$$
$$= 1 - 2 + 2 = \boxed{1}$$

3. $\int_1^8 x^{-2/3} \mathrm{d}x$

$$\int_1^8 x^{-2/3} dx = 3x^{1/3}\Big|_1^8$$
$$= 3\left(\sqrt[3]{8} - \sqrt[3]{1}\right)$$
$$= 3(2-1)$$
$$= \boxed{3}$$

4. $\int_{\pi/6}^{\pi/2} \csc(t)\cot(t)\,\mathrm{d}t$

$$\int_{\pi/6}^{\pi/2} \csc(t)\cot(t) = -\csc(t)\Big|_{\pi/6}^{\pi/2}$$
$$= -\left(\csc(\pi/2) - \csc(\pi/6)\right)$$
$$= \csc(\pi/6) - \csc(\pi/2)$$
$$= \frac{1}{\sin(\pi/6)} - \frac{1}{\sin(\pi/2)}$$
$$= \frac{1}{1/2} - 1 = 2 - 1 = \boxed{1}$$

Date: January 28, 2019.

5. $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2(\theta)\,\mathrm{d}\theta$

$$\int_{\pi/4}^{\pi/3} \csc^2(\theta)\,d\theta = -\cot(\theta)\Big|_{\pi/4}^{\pi/3}$$
$$= \cot(\pi/4) - \cot(\pi/3)$$
$$= 1 - \frac{\cos(\pi/3)}{\sin(\pi/3)}$$
$$= 1 - \frac{1/2}{\sqrt{3}/2}$$
$$= 1 - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) = \boxed{1 - \frac{\sqrt{3}}{3}}$$

6. $\displaystyle\int_0^{\pi/4} \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$

$$\int_0^{\pi/4} \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta)\Big|_0^{\pi/4}$$
$$= \sec(\pi/4) - \sec(0)$$
$$= \frac{1}{\cos(\pi/4)} - \frac{1}{\cos(0)}$$
$$= \frac{1}{\frac{\sqrt{2}}{2}} - \frac{1}{1}$$
$$= \frac{2}{\sqrt{2}} - 1$$
$$= \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} - 1$$
$$= \frac{2\sqrt{2}}{2} - 1 = \boxed{\sqrt{2} - 1}$$

7. $\displaystyle\int_0^1 \cos(\pi t/2)\,\mathrm{d}t$

$u = \frac{\pi t}{2}$, $du = \frac{\pi}{2}dt \Rightarrow \frac{2du}{\pi} = dt$

$$\int_0^1 \cos\left(\frac{\pi t}{2}\right)dt = \int_{u(0)}^{u(1)} \cos(u)\left(\frac{2}{\pi}\right)du$$
$$= \frac{2}{\pi}\int_0^{\pi/2} \cos(u)\,du$$
$$= \frac{2}{\pi}\left[\sin(\pi/2) - \sin(0)\right]$$
$$= \frac{2}{\pi}(1 - 0) = \boxed{\frac{2}{\pi}}$$

8. $\displaystyle\int_0^1 (2t-1)^{50}\,\mathrm{d}t$

$u = 2t - 1$
$du = 2dt \Rightarrow \frac{1}{2}du = dt$

$$\int_0^1 (2t-1)^{50}\,dt = \int_{u(0)}^{u(1)} u^{50}\frac{du}{2} = \frac{1}{2}\int_{-1}^{1} u^{50}\,du$$
$$= \frac{1}{2}\left(\frac{1}{51}\right)\left(1^{51} - (-1)^{51}\right)$$
$$= \frac{1}{2}\left(\frac{1}{51}\right)(1 - (-1)) = \frac{1}{2}\left(\frac{1}{51}\right)(2)$$
$$= \boxed{1/51}$$

9. $\displaystyle\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}\,\mathrm{d}t$

$u = \cos(t)$, $du = -\sin(t)dt \Rightarrow -du = \sin(t)dt$

$$\int_0^{\pi/6} \frac{\sin(t)}{\cos^2(t)}dt = \int_{u(0)}^{u(\pi/6)} \frac{-du}{u^2} = -\int_1^{\frac{\sqrt{3}}{2}} u^{-2}du = \frac{-1}{-1}u^{-1}\Big|_1^{\sqrt{3}/2}$$
$$= \left[\left(\frac{\sqrt{3}}{2}\right)^{-1} - 1\right] = \frac{2}{\sqrt{3}} - 1$$
$$= \boxed{\frac{2\sqrt{3}}{3} - 1}$$

10. $\displaystyle\int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x$

$u = 9 - x^2$, $du = -2x\,dx \Rightarrow \frac{-1}{2}du = x\,dx$

$$\int_0^3 \sqrt{9-x^2}\,x\,dx = \int_{u(0)}^{u(3)} \sqrt{u}\left(-\frac{1}{2}du\right)$$
$$= \frac{-1}{2}\int_9^0 u^{1/2}du = -\frac{1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|^0$$
$$= \frac{-1}{3}\left(0 - 9^{3/2}\right) = \frac{3^3}{3} = \boxed{9}$$

11. $\int_0^{\pi/2} \cos(x)\sin(\sin(x))\,\mathrm{d}x$

$u = \sin(x)$, $du = \cos(x)dx$

$$\int_0^{\pi/2} \cos(x)\sin(\sin(x))dx = \int_{u(0)}^{u(\pi/2)} \sin(u)du$$

$$= \int_0^1 \sin(u)du$$

$$= -\cos(u)\Big|_0^1$$

$$= -(\cos(1) - \cos(0))$$

$$= \boxed{1 - \cos(1)}$$

12. $\int_{-\pi/3}^{\pi/3} x^4 \sin(x)\,\mathrm{d}x$

$f(x) = x^4\sin(x)$, $f(-x) = (-x)^4\sin(-x)$

$= x^4(-\sin(x))$

$= -x^4\sin(x) = -f(x)$ is odd

Let $u = -x$, so $-du = dx$

$$\Rightarrow \int_{-\pi/3}^{0} f(x)dx = \int_{u(-\pi/3)}^{u(0)} f(-u)(-du)$$

$$= \int_{\pi/3}^{0} f(u)du$$

$$= -\int_0^{\pi/3} f(u)du = -\int_0^{\pi/3} f(x)dx$$

$$\Rightarrow \int_{-\pi/3}^{\pi/3} f(x)dx = \int_{-\pi/3}^{0} f(x)dx + \int_{\pi/3}^{0} f(x)dx = \boxed{0}$$