APPROXIMATE INTEGRATION

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Name: Solutions

In the following Problems, we will estimate

$$\int_0^\pi \sin^2(x)\,dx = \frac{\pi}{2}.$$

1. Find the number of intervals required to estimate the integral using the Trapezoid or Midpoint rules with an error of no more than 10^{-4}.

$f(x) = \sin^2(x)$

$f'(x) = 2\sin(x)\cos(x)$

$f''(x) = 2(\cos^2(x) - \sin^2(x))$

$f'''(x) = 2(-2\cos(x)\sin(x) - 2\sin(x)\cos(x))$
$= -8\cos(x)\sin(x)$

On $[0,\pi]$

$f'''(x) = 0 \iff \cos(x) = 0$ or $\sin(x) = 0$
$\iff x = 0,\ x = \pi/2,$ or $x = \pi$

$$\left.\begin{aligned} f''(0) &= 2(1-0) = 2 \\ f''(\pi/2) &= 2(0-1) = -2 \\ f''(\pi) &= 2(1-0) = 2 \end{aligned}\right\} |f''(x)| \leq 2$$

$$|E_T| < \frac{2\pi^3}{12n^2} = \frac{\pi^3}{6n^2} < \frac{1}{10^4}$$

$$\Rightarrow n > \sqrt{\frac{10^4\pi^3}{6}} \approx 227.3$$

So we need at least 228 intervals.

$$|E_M| < \frac{2\pi^3}{24n^2} = \frac{\pi^3}{12n^2} < \frac{1}{10^4}$$

$$\Rightarrow n > \sqrt{\frac{10^4\pi^3}{12}} \approx 160.7$$

So we need at least 161 intervals.

2. Find the number of intervals required to estimate the integral using Simpson's Rule to with an error of no more than 10^{-4}.

$$f^{IV}(x) = -8\left(\cos^2(x) - \sin^2(x)\right)$$

$$f^{V}(x) = -8\left(-2\cos(x)\sin(x) - 2\sin(x)\cos(x)\right)$$

$$= -32\cos(x)\sin(x).$$

$$= 0 \iff x = 0, \pi/2, \text{ or } \pi$$

$$\left.\begin{array}{l} f^{IV}(0) = -8(1-0) = -8 \\ f^{IV}\left(\frac{\pi}{2}\right) = -8(0-1) = 8 \\ f^{IV}(\pi) = -8(1-0) = -8 \end{array}\right\} \Rightarrow |f^{IV}(x)| \leq 8 \text{ on } [0,\pi]$$

$$|E_S| < \frac{8\pi^5}{180n^4} = \frac{2\pi}{45n^4} < \frac{1}{10^4}$$

$$\Rightarrow n > \sqrt[4]{\frac{10^4(2\pi^5)}{45}} \approx 19.2$$

So we need at least 20 intervals.

3. Estimate the value of the integral using M_2, T_2, and S_4. For each of these, what is the error from your estimate?

$n=2$

$$\Delta x = \frac{\pi - 0}{2} = \frac{\pi}{2}$$

$$x_0 = 0,\quad x_1 = \frac{\pi}{2},\quad x_2 = \pi$$

$$\bar{x}_1 = \frac{0+\frac{\pi}{2}}{2} = \frac{\pi}{4},\quad \bar{x}_2 = \frac{\frac{\pi}{2}+\pi}{2} = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$$

$$M_2 = \left[\sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{3\pi}{4}\right)\right]\frac{\pi}{2}$$

$$= \left(\frac{2}{4} + \frac{2}{4}\right)\frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

$$T_2 = \frac{\left[\sin^2(0) + 2\sin^2\left(\frac{\pi}{2}\right) + \sin^2(\pi)\right]}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{0+2+0}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{2}}$$

$$S_4 = S_{2(2)} = \frac{2M_2 + T_2}{3} = \frac{2(\pi/2) + \frac{\pi}{2}}{3} = \frac{3\left(\frac{\pi}{2}\right)}{3} = \boxed{\frac{\pi}{2}}$$