AREA BETWEEN CURVES AND VOLUME

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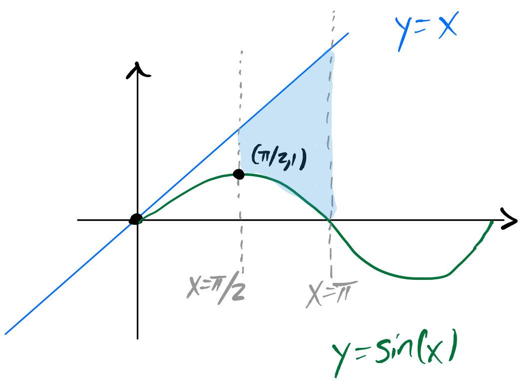
Name: Solutions

Area Between Curves

For each problem, sketch the region enclosed by the two curves and compute its area.

1. $y = \sin(x)$, $y = x$, $x = \pi/2$, $x = \pi$



$$\int_{\pi/2}^{\pi} x - \sin(x)\,dx = \frac{1}{2}\left((\pi)^2 - \frac{\pi^2}{4}\right) + \left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{4\pi^2 - \pi^2}{4}\right) + (-1 - 0)$$

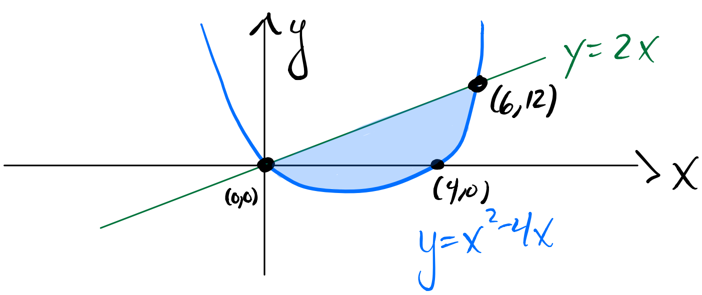
$$= \boxed{\frac{3\pi^2}{8} - 1}$$

2. $y = x^2 - 4x$, $y = 2x$

$$x^2 - 4x = 2x \iff x^2 - 6x = 0$$

$$\iff x(x-6) = 0$$

$$\iff x = 0 \text{ or } x = 6$$



$$A = \int_0^6 2x - (x^2 - 4x)\,dx$$

$$= 6\int_0^6 x\,dx - \int_0^6 x^2\,dx$$

$$= 6\left(\frac{1}{2}\right)(6^2 - 0^2) - \frac{1}{3}(6^3 - 0^3)$$

$$= 3(36) - 2(36)$$

$$= (3-2)(36) = \boxed{36}$$

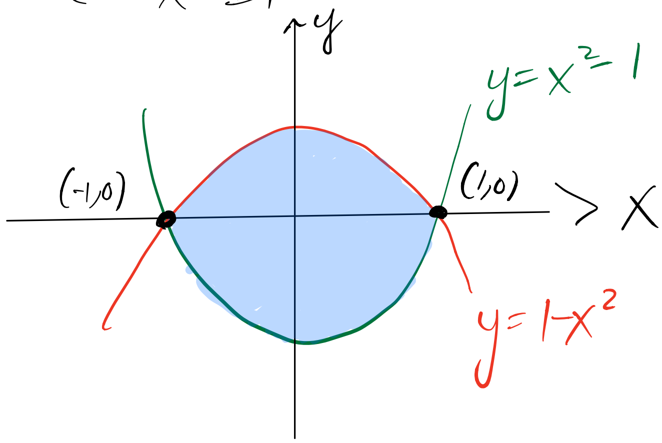
3. $y = 1 - x^2$, $y = x^2 - 1$

$1 - x^2 = x^2 - 1$

$\iff 2 = 2x^2$

$\iff x^2 = 1$

$\iff x = \pm 1$



$$A = \int_{-1}^{1} (1-x^2) - (x^2 - 1)\,dx$$

$$= \int_{-1}^{1} 2 - 2x^2\,dx$$

$$= 2\left[\int_{-1}^{1} dx - \int_{-1}^{1} x^2\,dx\right]$$

$$= 2\left(1 - (-1) - \tfrac{1}{3}\left(1^3 - (-1)^3\right)\right)$$

$$= 2\left(2 - \tfrac{1}{3}(2)\right)$$

$$= 4\left(1 - \tfrac{1}{3}\right)$$

$$= 4\left(\tfrac{2}{3}\right) = \boxed{\tfrac{8}{3}}$$

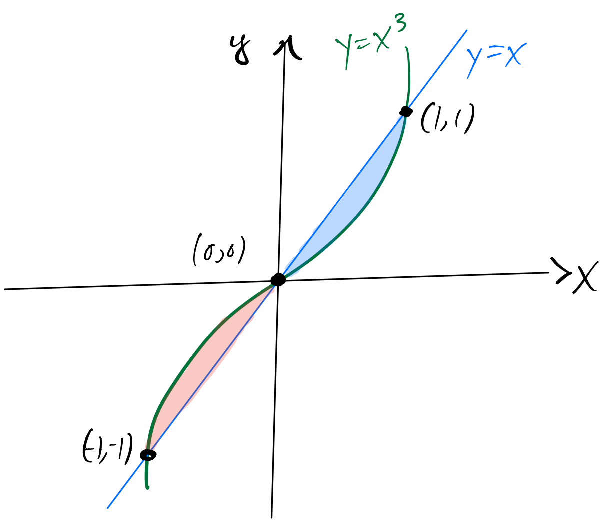
4. $y = x^3$, $y = x$

$x^3 = x \iff x^3 - x = 0$

$\iff x(x^2 - 1) = 0$

$\iff x(x+1)(x-1) = 0$

$\iff x = 0,\ x = 1,\ \text{or } x = -1.$



$$A = \int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx$$

$$= \tfrac{1}{4}x^4\Big|_{-1}^{0} - \tfrac{1}{2}x^2\Big|_{-1}^{0} + \tfrac{1}{2}x^2\Big|_{0}^{1} - \tfrac{1}{4}x^4\Big|_{0}^{1}$$

$$= \tfrac{1}{4}(0-1) - \tfrac{1}{2}(0-1) + \tfrac{1}{2}(1-0) - \tfrac{1}{4}(1-0)$$

$$= \tfrac{-1}{4} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{4}$$

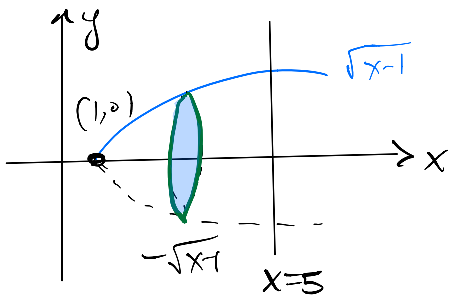
$$= 1 - \tfrac{1}{2}$$

$$= \boxed{\tfrac{1}{2}}$$

Volumes

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the curves and a typical cross section of the solid.

5. $y = \sqrt{x-1}$, $y = 0$, $x = 1$, $x = 4$; about the x-axis.



$$V = \int_1^4 A(x)\,dx$$

$$= \pi \int_1^4 x-1\,dx$$

$u = x-1$

$du = dx$

$u(4) = 4-1 = 3$

$u(1) = 1-1 = 0$

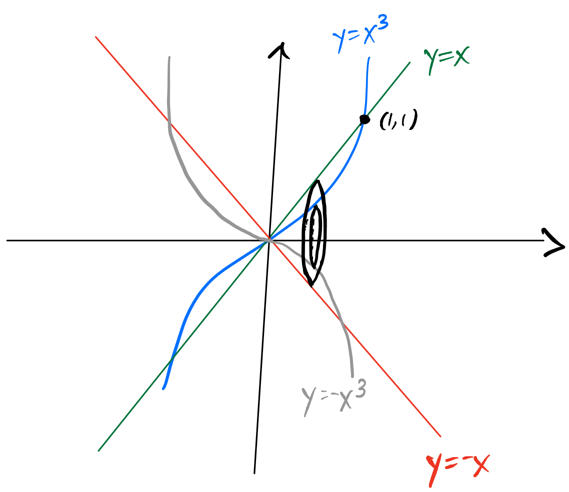
$$= \pi \int_0^3 u\,du$$

$$= \frac{\pi}{2} u^2 \Big|_0^3$$

$$= \boxed{\frac{9\pi}{2}}$$

$$A(x) = A(\quad) = \pi r^2 = \pi(x-1)$$

6. $y = x^3$, $y = x$, $0 \leq x$; about the x-axis



$$V = \int_0^1 A(x)\,dx$$

$$= \pi \int_0^1 x^2 - x^6\,dx$$

$$= \pi\left(\frac{1}{3}x^3\Big|_0^1 - \frac{1}{7}x^7\Big|_0^1\right)$$

$$= \pi\left(\frac{1}{3}(1-0) - \frac{1}{7}(1-0)\right)$$

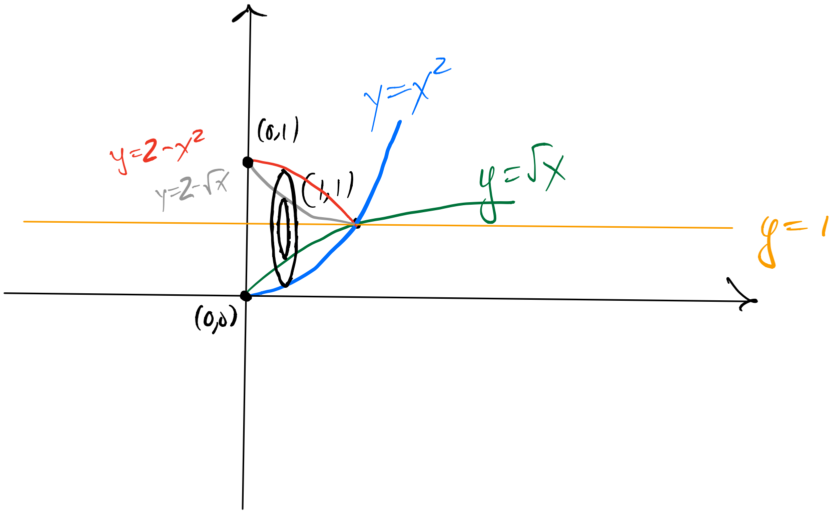
$$= \pi\left(\frac{7-3}{21}\right)$$

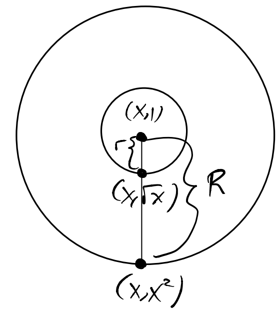
$$= \boxed{\frac{4\pi}{21}}$$

$$A(x) = A(\quad) = \pi x^2 - \pi(x^3)^2$$

$$= \pi(x^2 - x^6)$$

7. $y = x^2$, $x = y^2$; about $y = 1$





$R = 1 - x^2, \quad r = 1 - \sqrt{x}$

$$A(x) = A(\text{◎}) = \pi(1-x^2)^2 - \pi(1-\sqrt{x})^2$$

$$= \pi\left(1 - 2x^2 + x^4 - (1 - 2\sqrt{x} + x)\right)$$

$$= \pi(1 - 2x^2 + x^4 - 1 + 2\sqrt{x} - x)$$

$$= \pi(x^4 - 2x^2 - x + 2x^{1/2})$$

$$V = \int_0^1 A(x)\,dx = \pi\left(\int_0^1 x^4\,dx - 2\int_0^1 x^2\,dx - \int_0^1 x\,dx + 2\int_0^1 x^{1/2}\,dx\right)$$

$$= \pi\left(\tfrac{1}{5}(1^5 - 0^5) - 2\left(\tfrac{1}{3}\right)(1^3 - 0^3) - \tfrac{1}{2}(1^2 - 0^2) + 2\left(\tfrac{2}{3}\right)(1^{3/2} - 0^{3/2})\right)$$

$$= \pi\left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right)$$

$$= \pi\left(\frac{1}{5} - \frac{1}{2} + \frac{2}{3}\right)$$

$$= \pi\left(\frac{6 - 15 + 20}{30}\right)$$

$$= \boxed{\frac{11\pi}{30}}$$