GEOMETRIC SERIES

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Name: Solutions

Theorem. *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

converges if $|r| < 1$ *and diverges otherwise. The sum of the convergent series is*

$$s = \frac{a}{1-r}, \ |r| < 1.$$

Observation: For a geometric series, we can always find the value of r by taking the ratio of any two consecutive terms:

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r.$$

1. Determine whether the series

$$4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$$

converges or diverges. If it is convergent, find its sum.

$a = 4, \quad r = \frac{3}{4}$

$$4 + 3 + \frac{9}{4} + \frac{27}{16} = \sum_{n=1}^{\infty} 4\left(\frac{3}{4}\right)^{n-1} = \frac{4}{1 - 3/4} = \frac{4}{1/4} = \boxed{16}$$

2. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k(k+14)}{(k+15)^2}$$

converges or diverges. If it converges, find its sum.

$$\lim_{k\to\infty} \frac{k(k+14)}{(k+15)^2} \overset{L'H}{=} \lim_{k\to\infty} \frac{(k+4)+k}{2(k+15)}$$

$$\overset{L'H}{=} \lim_{k\to\infty} \frac{2}{2} = 1 \neq 0$$

So $\sum_{k=1}^{\infty} \frac{k(k+14)}{(k+15)^2}$ Diverges by the n^{th} Term Test for Divergence.

3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$$

converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{2^n+1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$$

$$= \frac{\frac{2}{3}}{1-\frac{2}{3}} + \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{2}{3-2} + \frac{1}{3-1} = \boxed{5/2}$$

4. Express $0.\bar{8}$ as a rational number (i.e. a ratio of two integers).

$$0.\bar{8} = \frac{8}{10} + \frac{8}{10^2} + \frac{8}{10^3} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{8}{10}\left(\frac{1}{10}\right)^{n-1}$$

$$= \frac{\frac{8}{10}}{1-\frac{1}{10}} = \frac{8}{10-1} = \boxed{\frac{8}{9}}$$

5. Express $2.\overline{516}$ as a rational number.

$$2.\overline{516} = 2 + 0.\overline{516}$$

$$= 2 + \frac{516}{10^3} + \frac{516}{10^6} + \frac{516}{10^9} + \cdots$$

$$= 2 + \sum_{n=1}^{\infty} \frac{516}{10^3}\left(\frac{1}{10^3}\right)^{n-1}$$

$$= 2 + \frac{\frac{516}{10^3}}{1-\frac{1}{10^3}}$$

$$\begin{array}{r} 83 \\ 3\overline{)2514} \\ 24 \\ \hline 11 \end{array}$$

$$= 2 + \frac{516}{10^3 - 1}$$

$$= \frac{2(999)+516}{999} = \frac{2514}{999} = \frac{3(838)}{3(333)}$$

$$= \boxed{\frac{838}{333}}$$