IMPROPER INTEGRALS

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Name: Solutions

Determine whether the following improper integrals converge or diverge.

Type I

1. $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,\mathrm{d}x$

Let $u = \ln(x)$, $du = \frac{1}{x}dx$, so

$$\int_1^\infty \frac{\ln(x)}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x}dx = \lim_{t\to\infty}\int_0^{\ln(t)} u\,du$$

$$= \lim_{t\to\infty}\frac{1}{2}u^2\Big|_0^{\ln(t)} = \lim_{t\to\infty}\frac{1}{2}\ln(t)^2 = \boxed{\infty}$$

2. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x$

Let $u = 2x+1$, $du = 2dx \Rightarrow \frac{1}{2}du = dx$ $\quad u(1) = 2(1)+1 = 3$

$u(t) = 2t+1$

$$\int_1^\infty \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty}\frac{1}{2}\int_3^{2t+1} u^{-3}du$$

$$= \lim_{t\to\infty}\frac{1}{2}\left(\frac{-1}{2}\right)u^{-2}\Big|_3^{2t+1} = \lim_{t\to\infty}\frac{-1}{4}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right]$$

$$= \frac{-1}{4}\left[0 - \frac{1}{9}\right] = \boxed{\frac{1}{36}}$$

3. $\int_{-\infty}^{0} e^x \, dx$

$$\int_{-\infty}^{0} e^x dx = \lim_{t\to\infty} \int_{-t}^{0} e^x dx = \lim_{t\to\infty} e^x \Big|_{-t}^{0} = \lim_{t\to\infty} e^0 - e^{-t}$$

$$= \lim_{t\to\infty} 1 - \frac{1}{e^t} = \boxed{1}$$

4. $\int_{-\infty}^{\infty} xe^{-x^2} \, dx$

Let $u = -x^2$, so $du = -2x dx \Rightarrow \frac{-1}{2} du = x dx$

$$\int_{0}^{\infty} xe^{-x^2} dx = \lim_{t\to\infty} \int_{0}^{t} xe^{-x^2} dx = \lim_{t\to\infty} \frac{-1}{2} \int_{0}^{-t^2} e^u du$$

$$= \lim_{t\to\infty} \frac{-1}{2} e^u \Big|_{0}^{-t^2} = \lim_{t\to\infty} \frac{-1}{2} \left[\frac{1}{e^{t^2}} - 1 \right] = \frac{1}{2}$$

$$\int_{-\infty}^{0} xe^{-x^2} dx = \lim_{t\to\infty} \int_{-t}^{0} xe^{-x^2} dx = \lim_{t\to\infty} \frac{-1}{2} \int_{-t^2}^{0} e^u du$$

$$= \lim_{t\to\infty} \frac{-1}{2} e^u \Big|_{-t^2}^{0} = \lim_{t\to\infty} \frac{-1}{2} \left[1 - \frac{1}{e^{t^2}} \right] = \frac{-1}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} xe^{-x^2} dx = \frac{1}{2} + \left(\frac{-1}{2}\right) = \boxed{0}$$

Type II

5. $\int_0^1 \ln(x)\,\mathrm{d}x$

$$\int_0^1 \ln(x)dx = \lim_{t\to 0^+} \int_t^1 \ln(x)dx = \lim_{t\to 0^+} x\ln(x)\Big|_t^1 - x\Big|_t^1$$

$$= \lim_{t\to 0^+} \left[(0 - t\ln(t)) - (1-t)\right]$$

$$= \lim_{t\to 0^+} t-1-t\ln(t) = \boxed{-1}$$

because

$$\lim_{t\to 0^+} -t\ln(t) = \lim_{t\to 0^+} \frac{-\ln(t)}{\left(\frac{1}{t}\right)} \overset{L'H}{=} \lim_{t\to 0^+} \frac{-\left(\frac{1}{t}\right)}{\left(\frac{-1}{t^2}\right)} = \lim_{t\to 0^+} \frac{t^2}{t} = \lim_{t\to 0^+} t = 0.$$

6. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1$$

$$= \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2-0 = \boxed{2}$$