# PARAMETRIC EQUATIONS


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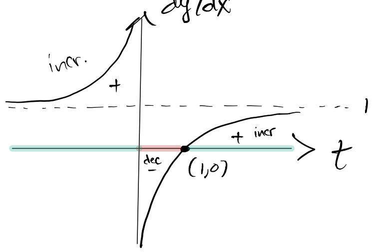
Name: Solutions

Sketch the curve by using parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as $t$ increases.

**1.** $x = 1 - t^2$, $y = 2t - t^2$, $-1 \le t \le 2$.

$$\frac{dy}{dx} = \frac{2-2t}{-2t} = \frac{-(1-t)}{t} = \frac{t-1}{t} = 1 - \frac{1}{t}$$




$$\frac{d^2y}{dx^2} = \frac{t^{-2}}{-2t} = \frac{-1}{2t^3} \begin{cases} < 0 & t > 0 \\ > 0 & t < 0 \end{cases}$$

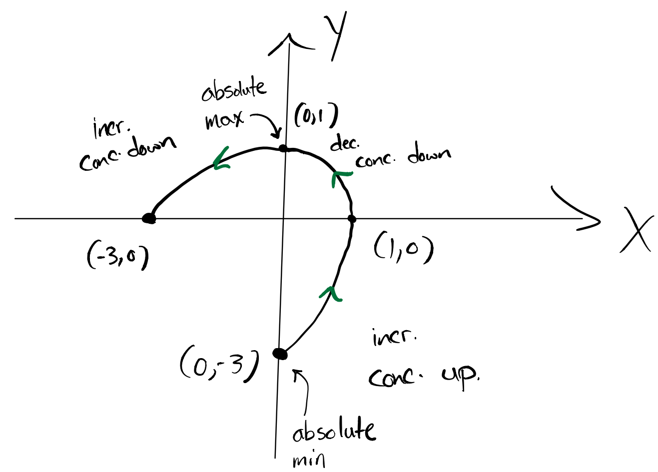
$x = (1-t)(1+t)$
$y = t(2-t)$

$\underline{t=-1}$ $x=0$, $y=-3$

$\underline{t=0}$ $x=1$, $y=0$

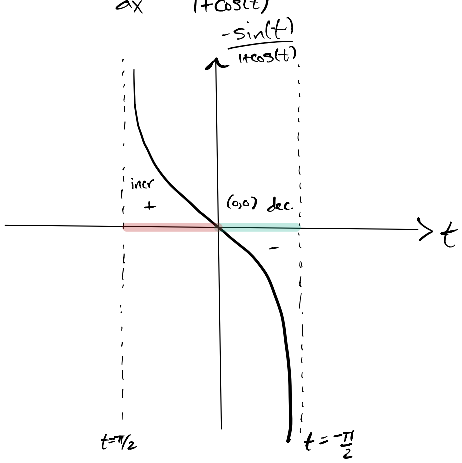
$\underline{t=1}$ $x=0$, $y=1$

$\underline{t=2}$ $x=-3$, $y=0$.




**2.** $x = t + \sin(t)$, $y = \cos(t)$, $-\pi \le t \le \pi$.

$$\frac{dy}{dx} = \frac{-\sin(t)}{1+\cos(t)}$$




$$\frac{d}{dt}\frac{\sin(t)}{1+\cos(t)} = \frac{\cos(t)(1+\cos(t)) - \sin(t)(-\sin(t))}{(1+\cos(t))^2}$$

$$= \frac{\cos(t) + \cos^2(t) + \sin^2(t)}{(1+\cos(t))^2}$$

$$= \frac{\cos(t)+1}{(1+\cos(t))^2}$$

$$= \frac{1}{1+\cos(t)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{d}{dt}\frac{\sin(t)}{1+\cos(t)}}{1+\cos(t)} = \left(\frac{-1}{1+\cos(t)}\right)\left(\frac{1}{1+\cos(t)}\right) = \frac{-1}{(1+\cos(t))^2}$$

$\Rightarrow$ Concave down on $(-\pi, \pi)$

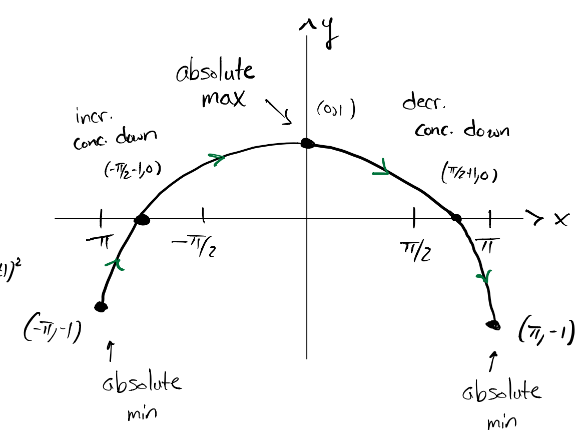
$\underline{t=-\pi}$ $x=-\pi$, $y=-1$

$\underline{t=\frac{\pi}{2}}$ $x=\pi/2+1$, $y=0$

$\underline{t=-\frac{\pi}{2}}$ $x=-\frac{\pi}{2}-1$, $y=0$

$\underline{t=\pi}$ $x=\pi$, $y=-1$

$\underline{t=0}$ $x=0$, $y=1$



**3.** Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve and indicate the direction in which the curve is traced as the parameter increases.

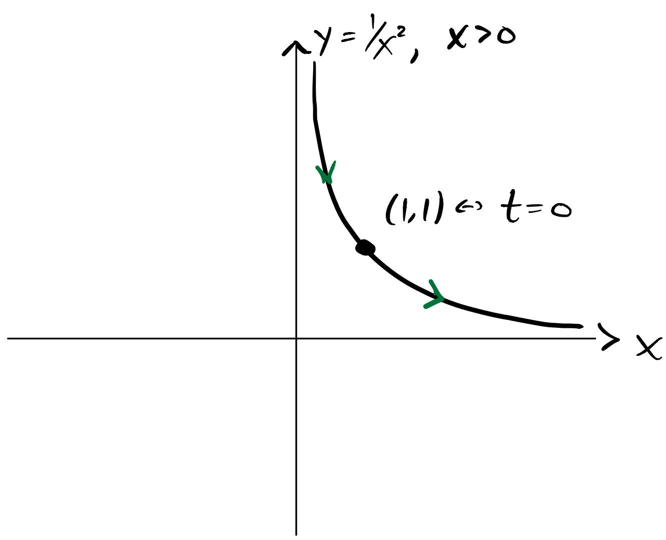
$$x = e^t,\ y = e^{-2t}.$$

$$y = e^{-2t} = (e^t)^{-2} = x^{-2} = \frac{1}{x^2}$$

Note that

- $x = e^t > 0$ for all $t$
- $x \to 0$ as $t \to -\infty$
- $x \to \infty$ as $t \to \infty$

So we obtain the right side of $y = \frac{1}{x^2}$



**4.** Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$x = \sqrt{t},\ y = t^2 - 2t,\ t = 4.$$

$$\frac{dy}{dx} = \frac{2t-2}{\frac{1}{2}t^{-1/2}} = \frac{2(t-1)}{\frac{1}{2\sqrt{t}}} = 2(t-1)(2\sqrt{t}) = 4(t-1)\sqrt{t}$$

$$\frac{dy}{dx}(4) = 4(4-1)\sqrt{4} = 12(2) = 24$$

$$x(4) = \sqrt{4} = 2,\ y(4) = 4^2 - 2(4) = 16 - 8 = 8$$

$$\boxed{y - 8 = 24(x-2)}$$

**5.** Use the formula

$$L = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

to find the length of the curve

$$x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 2$$

$$\frac{dx}{dt} = e^t - 1, \quad \frac{dy}{dt} = 4\left(\tfrac{1}{2}\right)e^{t/2} = 2\sqrt{e^t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{e^{2t} - 2e^t + 1 + 4e^t}$$
$$= \sqrt{e^{2t} + 2e^t + 1}$$
$$= \sqrt{(e^t+1)^2}$$
$$= e^t + 1.$$

$$L = \int_0^2 (e^t + 1)\,dt$$
$$= e^t\Big|_0^2 + t\Big|_0^2$$
$$= (e^2 - 1) + (2 - 0)$$
$$= \boxed{e^2 + 1}$$

**6.** Use the formula

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

to find the area of the surface obtained by rotating the curve about the $x$-axis

$$x = t^3,\ y = t^2,\ 0 \le t \le 1$$

$$\left(\frac{dx}{dt}\right)^2 = (3t^2)^2 = 9t^4 \ ; \quad \left(\frac{dy}{dt}\right)^2 = (2t)^2 = 4t^2$$

$$S = \int_0^1 2\pi(t^2)\sqrt{9t^4 + 4t^2}\,dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{t^2(9t^2+4)}\,dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{9t^2+4}\; t\,dt$$

$u = 9t^2+4 \iff t^2 = \frac{u-4}{9}$ $\quad u(0) = 4$

$du = 18t\,dt$ $\quad u(1) = 9+4 = 13$

$\Rightarrow \frac{1}{18}du = t\,dt$

$$= 2\pi \int_4^{13} \left(\frac{u-4}{9}\right) u^{1/2}\left(\frac{1}{18}du\right)$$

$$= \frac{2\pi}{9(18)} \int_4^{13} u^{3/2} - 4u^{1/2}\,du$$

$$= \frac{\pi}{81}\left[\int_4^{13} u^{3/2}du - 4\int u^{1/2}du\right]$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^{5/2}\Big|_4^{13} - 4\left(\frac{2}{3}\right)u^{3/2}\Big|_4^{13}\right]$$

$$= \frac{\pi}{81}\left[\frac{2}{5}\left(13^{5/2} - 2^5\right) - \frac{8}{3}\left(13^{3/2} - 2^3\right)\right]$$

$$= \frac{\pi}{81}\left[\frac{2}{5}\left(13^2\sqrt{13} - 32\right) - \frac{8}{3}\left(13\sqrt{13} - 8\right)\right]$$

$$= \frac{\pi}{81}\left[\frac{6(13^2\sqrt{13} - 32) - 40(13\sqrt{3} - 8)}{15}\right]$$

$$= \frac{\pi}{81}\left[\frac{6(13^2)\sqrt{13} - 192 - 40(13\sqrt{3}) + 320}{15}\right]$$

$$= \frac{\pi}{81}\left[\frac{6(13^2)\sqrt{13} - 40(13)(\sqrt{13}) + 128}{15}\right]$$

$$= \frac{\pi}{81}\left(\frac{13\sqrt{13}(78 - 40) + 128}{15}\right)$$

$$= \boxed{\frac{\pi(494\sqrt{13} + 128)}{1215} \approx 4.9364\ldots}$$