PARTIAL FRACTIONS

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Name: Solutions

Evaluate the following integrals

Case I

1. $\displaystyle\int \frac{x^4}{x-1}\,\mathrm{d}x$

$$\begin{array}{r|l} & x^3+x^2+x+1 \\ \hline x-1 & x^4 \\ & -x^4+x^3 \\ \hline & x^3 \\ & -x^3+x^2 \\ \hline & x^2 \\ & -x^2+x \\ \hline & x \\ & -x+1 \\ \hline & 1 \end{array}$$

$$\int \frac{x^4}{x-1}\,dx = \int\left[x^3+x^2+x+1+\frac{1}{x-1}\right]dx$$

$$= \boxed{\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C}$$

2. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,\mathrm{d}x$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$\Rightarrow 5x+1 = A(x-1) + B(2x+1)$$
$$= (A+2B)x - A + B$$

$$\begin{aligned} 5 &= A+2B \\ + \quad 1 &= -A+B \\ \hline 6 &= 0+3B \end{aligned}$$

$$\Rightarrow B = 6/3 = 2$$

$$A = B-1 = 2-1 = 1$$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$$
$$= \boxed{\tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C}$$

<u>Check</u>: $\dfrac{d}{dx}\left(\tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C\right)$

$$= \frac{1}{2}\left(\frac{2}{2x+1}\right) + 2\left(\frac{1}{x-1}\right)$$
$$= \frac{(x-1) + 2(2x+1)}{(2x+1)(x-1)}$$
$$= \frac{x-1+4x+2}{(2x+1)(x-1)} = \frac{5x+1}{(2x+1)(x-1)}$$

3. $\int \frac{2}{(2x+1)(x+1)}\,\mathrm{d}x$

$$\frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$\Rightarrow 2 = A(x+1) + B(2x+1) = (A+2B)x + A+B$$

$$\Rightarrow 0 = A+2B$$
$$2 = A+B$$

$$2 = 2-0 = (A+B) - (A+2B) = -B$$
$$\Rightarrow B = -2$$
$$2 = A-2 \Rightarrow A = 4$$

$$\int \frac{2}{(2x+1)(x+1)}\,dx = \int \frac{4}{2x+1}\,dx - \int \frac{2}{x+1}\,dx$$
$$= 4\,\frac{\ln|2x+1|}{2} - 2\ln|x+1| + C$$
$$= \boxed{2\ln|2x+1| - 2\ln|x+1| + C.}$$

<u>Check</u>: $\frac{d}{dx}\left(2\ln|2x+1| - 2\ln|x+1| + C\right)$

$$= 2\left(\frac{2}{2x+1}\right) - 2\left(\frac{1}{x+1}\right)$$
$$= \frac{4(x+1) - 2(2x+1)}{(2x+1)(x+1)} = \frac{4x+4-4x-2}{(2x+1)(x+1)} = \frac{2}{(2x+1)(x+1)}.$$

Case II

4. $\displaystyle\int \frac{2x+4}{(x-3)^2}\,\mathrm{d}x$

$$\frac{2x+4}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$$\Rightarrow 2x+4 = A(x-3) + B = Ax - 3A + B$$

$2 = A$

$4 = -3A + B = -6 + B \Rightarrow B = 10$

$$\int \frac{2x+4}{(x-3)^2}\,dx = \int \frac{2}{x-3}\,dx + \int \frac{10}{(x-3)^2}\,dx$$

$$= \boxed{2\ln|x-3| - \frac{10}{x-3} + C}$$

Check: $\displaystyle\frac{d}{dx}\left(2\ln|x-3| - \frac{10}{x-3} + C\right) = 2\left(\frac{1}{x-3}\right) + \frac{10}{(x-3)^2}$

$$= \frac{2(x-3)+10}{(x-3)^2}$$

$$= \frac{2x+4}{(x-3)^2}$$

Case III

5. $\displaystyle\int \frac{6x^2-2}{(x+1)(x-1)(x^2+1)}\,\mathrm{d}x$

$$\frac{6x^2-2}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow 6x^2-2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$$
$$= A(x^3-x^2+x-1) + B(x^3+x^2+x+1) + (Cx+D)(x^2-1)$$
$$= A(x^3-x^2+x-1) + B(x^3+x^2+x+1) + Cx^3+Dx^2-Cx-D$$
$$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

$0 = A+B+C$

$6 = -A+B+D$

$0 = A+B-C$

$-2 = -A+B-D$

$0 = 0-0 = (A+B+C)-(A+B-C) = 2C \Rightarrow C=0$

$6-(-2) = 8 = (-A+B+D)-(-A+B+D) = 2D \Rightarrow D=4$

$0 = A+B \Rightarrow -A = B$

$6 = -A+B+4 = 2B+4 \Rightarrow B = \frac{6-4}{2} = 1,\ A=-1$

$$\int \frac{6x^2-2}{(x+1)(x-1)(x^2+1)}\,dx = -\int\frac{dx}{x+1} + \int\frac{dx}{x-1} + 4\int\frac{dx}{x^2+1}$$
$$= \boxed{-\ln|x+1| + \ln|x-1| + 4\arctan(x) + C}$$

Check: $\displaystyle \frac{d}{dx}\left(-\ln|x+1| + \ln|x-1| + 4\arctan(x) + C\right) = \frac{-1}{x+1} + \frac{1}{x-1} + \frac{4}{x^2+1}$

$$= \frac{-(x-1)(x^2+1) + (x+1)(x^2+1) + 4(x+1)(x-1)}{(x+1)(x-1)(x^2+1)}$$

$$= \frac{-x^3+x^2-x+1+x^3+x^2+x+1+4x^2-4}{(x+1)(x-1)(x^2+1)} = \frac{6x^2-2}{(x+1)(x-1)(x^2+1)}$$

Case IV

6. $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

$$\frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$\Rightarrow x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + Cx+D = Ax^3 + (2A+B)x^2 + (2A+2B+C)x + (2B+D)$$

$$\Rightarrow 1 = A$$
$$2 = 2A+B = 2+B \Rightarrow B = 0$$
$$3 = 2A+2B+C = 2+C \Rightarrow C = 1$$
$$-2 = 2B+D = D$$

$$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int \frac{x}{x^2+2x+2}\,dx + \int \frac{x-2}{(x^2+2x+2)^2}\,dx$$

Write $x^2+2x+2 = (x+1)^2+1$ and make the substitution

$$u = x+1 \Leftrightarrow x = u-1, \quad du = dx$$

So we have the integrals

$$\int \frac{x}{x^2+2x+2}\,dx = \int \frac{u-1}{u^2+1}\,du = ① \int \frac{u}{u^2+1}\,du - ② \int \frac{1}{u^2+1}\,du$$

and

$$\int \frac{x-2}{(x^2+2x+2)^2}\,dx = \int \frac{u-3}{(u^2+1)^2}\,du = ③ \int \frac{u}{(u^2+1)^2}\,du - 3 ④ \int \frac{1}{(u^2+1)^2}\,du$$

Which we handle individually:

① Let $v = u^2+1$, so $\frac{1}{2}dv = \frac{1}{2}(2u\,du) = u\,du$ and

$$\int \frac{u}{u^2+1}\,du = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + C$$

$$= \frac{1}{2}\ln|u^2+1| + C = \frac{1}{2}\ln(x^2+2x+2) + C \qquad (x^2+2x+2>0)$$

② $\int \frac{1}{u^2+1}\,du = \arctan(u) + C = \arctan(x+1) + C$

③ Let $v = u^2+1$, so as above $\frac{1}{2}dv = u\,du$ and

$$\int \frac{u}{(u^2+1)^2}\,du = \frac{1}{2}\int \frac{dv}{v^2} = \frac{1}{2}\left(\frac{-1}{v}\right) + C = \frac{-1}{2v} + C$$

$$= \frac{-1}{2(u^2+1)} + C = \frac{-1}{2(x^2+2x+2)} + C$$

④ This requires a trigonometric substitution.

Let $u = \tan(\theta)$ so $du = \frac{d}{du}\tan(\theta)\,d\theta = \sec^2(\theta)\,d\theta$.

Next observe that

$$u^2+1 = \tan^2(\theta)+1 = \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta)$$

So

$$\int \frac{1}{(u^2+1)^2}\,du = \int \frac{\sec^2(\theta)}{(\sec^2(\theta))^2}\,d\theta$$

$$= \int \frac{\sec^2(\theta)}{\sec^4(\theta)}\,d\theta$$

$$= \int \frac{1}{\sec^2(\theta)}\,d\theta$$

$$= \int \cos^2(\theta)\,d\theta$$

$$= \int \left(\frac{1+\cos(2\theta)}{2}\right)d\theta$$

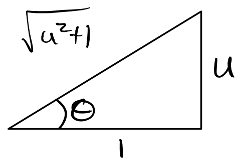
$$= \frac{1}{2}\left[\int d\theta + \int \cos(2\theta)\,d\theta\right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\left(2\sin(\theta)\cos(\theta)\right)\right] + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

Finally, build a triangle using $\frac{u}{1} = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{u}{\sqrt{u^2+1}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{u^2+1}}$$

So

$$\int \frac{1}{(u^2+1)^2}\,du = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

$$= \frac{1}{2}\arctan(u) + \frac{1}{2}\left(\frac{u}{\sqrt{u^2+1}}\right)\left(\frac{1}{\sqrt{u^2+1}}\right) + C$$

$$= \frac{1}{2}\arctan(u) + \frac{u}{2(u^2+1)} + C$$

$$= \frac{1}{2}\arctan(x+1) + \frac{x+1}{2(x^2+2x+2)} + C$$

Putting this all together yields

$$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \underbrace{\frac{1}{2}\ln|x^2+2x+2|}_{①} - \underbrace{\arctan(x+1)}_{②}$$

$$+ \underbrace{\frac{-1}{2(x^2+2x+2)}}_{③} - 3\left[\underbrace{\frac{1}{2}\arctan(x+1) + \frac{x+1}{2(x^2+2x+2)}}_{④}\right] + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\arctan(x+1) + \frac{-1-3(x+1)}{2(x^2+2x+2)} + C$$

$$= \boxed{\frac{1}{2}\ln(x^2+2x+1) - \frac{5}{2}\arctan(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C}$$