EXAM 1

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Date: September 25, 2019.

Inverse Trig Functions

1. Compute

$$\int \frac{\sin(x)}{1+\cos^2(x)}\,\mathrm{d}x$$

$u = \cos(x)$

$-du = \sin(x)dx$

$$\int \frac{\sin(x)}{1+\cos^2(x)}dx = \int \frac{-du}{1+u^2} = -\arctan(u)+C$$

$$= \boxed{-\arctan(\cos(x)) + C}$$

L'Hôpital's Rule

2. Compute
$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

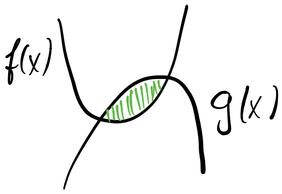
$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

$$\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \ln(x^{1/x})} = e^0 = \boxed{1}$$

Area and Volumes

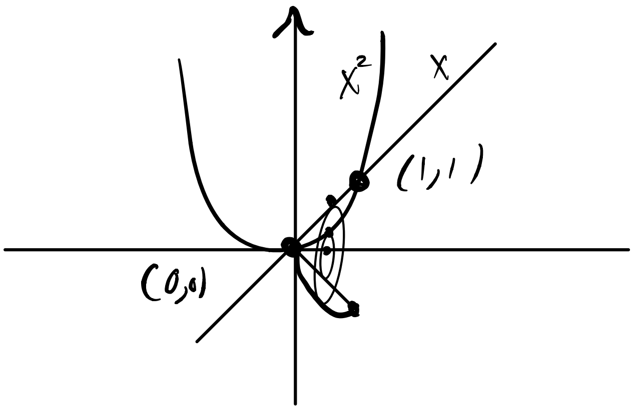
3. Compute the area of the region bounded by $f(x) = x^2 + 6$ and $g(x) = -2x^2 + 12x - 3$.



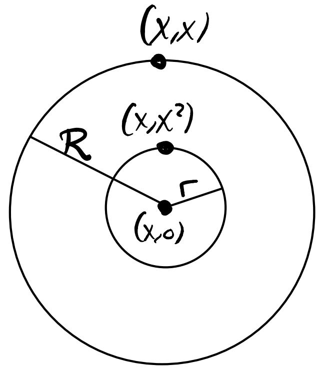
$$f(x) = g(x) \iff 0 = g(x) - f(x) = -2x^2 + 12x - 3 - (x^2+6)$$
$$= -3x^2 + 12x - 9$$
$$= -3(x^2 - 4x + 3)$$
$$= -3(x-1)(x-3)$$

$$\text{Area} = \int_1^3 [g(x) - f(x)]\,dx = -3\int_1^3 x^2\,dx + 12\int_1^3 x - 9\int_1^3 dx$$
$$= -x^3\Big|_1^3 + 6x^2\Big|_1^3 - 9x\Big|_1^3$$
$$= -(27-1) + 6(9-1) - 9(3-1)$$
$$= -26 + 48 - 18$$
$$= 22 - 18 = \boxed{4}$$

4. Compute the volume of the solid obtained by revolving the region bounded by $f(x) = x^2$ and $g(x) = x$ about the x-axis.



Cross-Section



$$A(x) = \pi R^2 - \pi r^2$$
$$= \pi x^2 - \pi x^4$$

$$\text{Volume} = \int_0^1 A(x)\,dx = \pi\left[\int_0^1 x^2\,dx - \int_0^1 x^4\,dx\right]$$
$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$
$$= \pi\left(\frac{5-3}{15}\right) = \boxed{\frac{2\pi}{15}}$$

Integration by Parts

5. Compute
$$\int \arctan(x)\,\mathrm{d}x$$

$u = \arctan(x)$ $\quad$ $v = x$

$du = \dfrac{dx}{1+x^2}$ $\quad$ $dv = dx$

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2}dx$$

$w = 1+x^2$

$\frac{1}{2}dw = xdx$

$$= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}dw$$

$$= x\arctan(x) - \frac{1}{2}\ln|w| + C$$

$$= \boxed{x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$$

Partial Fractions

6. Compute

$$\int \frac{3x+8}{(2x+3)(x+5)}\,\mathrm{d}x$$

$$\frac{3x+8}{(2x+3)(x+5)} = \frac{A}{2x+3} + \frac{B}{x+5}$$

$$\Rightarrow 3x+8 = A(x+5) + B(2x+3)$$

$$3(-5)+8 = -15+8 = -7 = B(-10+3) = -7B$$

$$\Rightarrow B = \frac{-7}{-7} = 1.$$

$$3\left(\frac{-3}{2}\right)+8 = \frac{-9}{2}+\frac{16}{2} = \frac{7}{2} = A\left(\frac{-3}{2}+\frac{10}{2}\right) = \frac{7A}{2}$$

$$\Rightarrow A=1$$

$$\int \frac{3x+8}{(2x+3)(x+5)}\,dx = \int \frac{1}{2x+3}\,dx + \int \frac{1}{x+5}\,dx$$

$$= \boxed{\frac{\ln|2x+3|}{2} + \ln|x+5| + C}$$

Approximate Integration

Use the function $f(x) = x^3 - x + 1$ to complete Problems 7 and 8.

7. Use the inequality

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

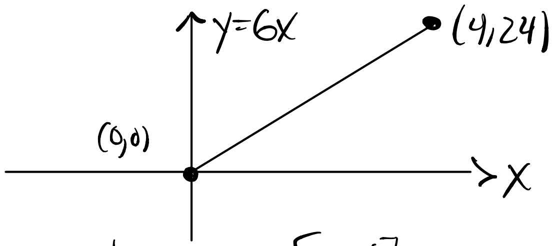
to determine how many intervals are needed to approximate

$$\int_0^4 f(x)\,\mathrm{d}x$$

using the Midpoint Rule with an error less than 10^{-2}.

$f'(x) = 3x^2 - 1$

$f''(x) = 6x$



$|f''(x)| \leq 6(4) = 24$ on $[0,4]$

So

$$|E_M| \leq \frac{24(4-0)^3}{24n^2} = \frac{64}{n^2} = \frac{8^2}{n^2} < \frac{1}{10^2}$$

$$\Rightarrow \quad 8^2 10^2 = (80)^2 < n^2$$

$$\Rightarrow \quad \boxed{80 < n}$$

8. Use the Midpoint Rule
$$M_n = \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$
with 2 intervals to estimate the value of the integral
$$\int_0^4 f(x)\,\mathrm{d}x.$$

$\Delta x = \frac{4-0}{2} = \frac{4}{2} = 2$

$x_0 = 0, \quad x_1 = 0+2 = 2, \quad x_2 = 2+2 = 4$

$\overline{x}_1 = \frac{0+2}{2} = 1, \quad \overline{x}_2 = \frac{2+4}{2} = 3$

$M_2 = f(1)\cdot 2 + f(3)\cdot 2$

$= (1^3 - 1 + 1)\cdot 2 + (3^3 - 3 + 1)\cdot 2$

$= 2 + (27 - 3 + 1)\cdot 2$

$= 2(1 + 27 - 3 + 1)$

$= 2(26)$

$= \boxed{52}$

Improper Integrals

9. Compute
$$\int_2^\infty \frac{2}{x^2-1}\,\mathrm{d}x$$

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\Rightarrow\ 2 = A(x+1) + B(x-1)$$
$$2 = A(-1+1) + B(-1-1) = -2B \Rightarrow B = -1$$
$$2 = A(1+1) + B(1-1) = 2A \Rightarrow A = 1$$

$$\int_2^\infty \frac{2}{x^2-1}dx = \lim_{t\to\infty} \int_2^t \frac{2}{x^2-1}dx$$
$$= \lim_{t\to\infty}\left[\int_2^t \frac{dx}{x-1} - \int_2^t \frac{dx}{x+1}\right]$$
$$= \lim_{t\to\infty}\left[\ln|x-1|\Big|_2^t - \ln|x+1|\Big|_2^t\right]$$
$$= \lim_{t\to\infty}\left[(\ln|t-1| - \ln|1|) - (\ln|t+1| - \ln(3))\right]$$
$$= \lim_{t\to\infty}\left[\ln\left|\frac{t-1}{t+1}\right| + \ln(3)\right]$$
$$= \ln\left(\lim_{t\to\infty}\frac{t-1}{t+1}\right) + \ln(3)$$
$$= \ln(1) + \ln(3) = \boxed{\ln(3)}$$