EXAM 1

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
By writing your name on the line below, you indicate that you have read and understand these directions.

Name: Solutions

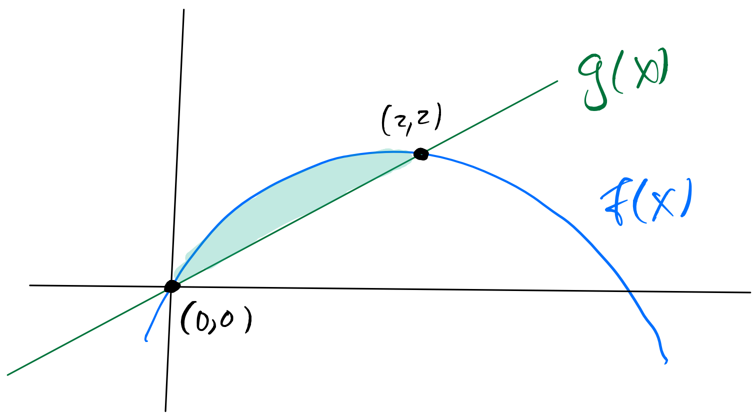
Problem	Points Earned	Points Possible
1		5
2		15
3		15
4		15
5		15
6		15
7		15
Total		100

Date: February 25, 2019.

Problems

In Problems 1-3, use the functions $f(x) = 3x - x^2$ and $g(x) = x$.

1. Sketch the area enclosed by f and g.



$3x - x^2 = x$

$\Leftrightarrow 2x - x^2 = 0$

$\Leftrightarrow x(2-x) = 0$

$\Leftrightarrow x = 0$ or $x = 2$

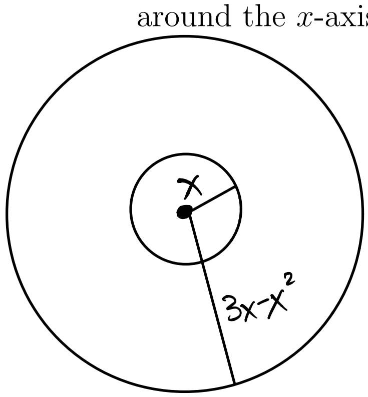
2. Find the area of the region from Problem 1.

$$\int_0^2 f(x) - g(x)\,dx = \int_0^2 3x - x^2 - x\,dx = \int_0^2 2x\,dx - \int_0^2 x^2\,dx$$

$$= x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2$$

$$= (4-0) - \frac{1}{3}(8-0) = \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

3. Sketch a typical cross section of the solid obtained by rotating the region in Problem 1 around the x-axis, then compute the volume of this solid.



$$A(x) = \pi(3x-x^2)^2 - \pi x^2 = \pi\left[9x^2 - 6x^3 + x^4 - x^2\right]$$

$$= \pi\left[x^4 - 6x^3 + 8x^2\right]$$

$$V = \int_0^2 A(x)\,dx = \pi\left[\int_0^2 x^4\,dx - 6\int_0^2 x^3\,dx + 8\int_0^2 x^2\,dx\right]$$

$$= \pi\left[\frac{1}{5}x^5\Big|_0^2 - \frac{6}{4}x^4\Big|_0^2 + \frac{8}{3}x^3\Big|_0^2\right]$$

$$= \pi\left[\frac{32}{5} - \frac{3}{2}(16) + \frac{8}{3}(8)\right]$$

$$= \pi\left(\frac{32}{5} - 24 + \frac{64}{3}\right) = \pi\left(\frac{96}{15} - \frac{360}{15} + \frac{320}{15}\right)$$

$$= \boxed{\pi\left(\frac{56}{15}\right)}$$

$24(15) = 240 + 100 + 20$
$= 360$

$64(5) = 300 + 20$
$= 320$

$96 - 40 = 56$

4. Compute $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$$

5. Compute $\int xe^{2x}\,\mathrm{d}x$

$u = x \qquad v = \frac{1}{2}e^{2x}$

$du = dx \qquad dv = e^{2x}\,dx$

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

6. Compute $\displaystyle\int \frac{\mathrm{d}x}{x^2-1}$.

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$\Rightarrow 1 = A(x-1) + B(x+1)$$
$$= (A+B)x + (-A+B)$$

$$\Rightarrow A+B=0$$
$$+ \quad \underline{-A+B=1}$$
$$0 + 2B = 1 \Rightarrow B = \frac{1}{2}, \; A = -B = \frac{-1}{2}$$

$$\int \frac{dx}{x^2-1} = \frac{-1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$= \frac{-1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$= \boxed{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

7. Use your answer to Problem 6 to decide whether the improper integral

$$\int_2^\infty \frac{\mathrm{d}x}{x^2-1}$$

converges or diverges. If it converges, find the value of the integral.

$$\int_2^\infty \frac{dx}{x^2-1} = \lim_{t\to\infty} \int_2^t \frac{dx}{x^2-1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \Bigg|_2^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln\left|\frac{t-1}{t+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right)$$

$$= \frac{1}{2}\left(\ln|1| - \ln\left|\frac{1}{3}\right|\right)$$

$$= \frac{-1}{2}\ln\left(\frac{1}{3}\right)$$

$$= -\frac{1}{2}(\ln(1) - \ln(3))$$

$$= -\frac{1}{2}(0 - \ln(3))$$

$$= \boxed{\frac{\ln(3)}{2}}$$