EXAM 2
MATH 161

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
By writing your name on the line below, you indicate that you have read and understand these directions.

Name: Solutions

Problem	Points Earned	Points Possible
1		5
2		4
3		2
4		1
5		3
6		15
7		20
8		15
9		15
10		20
Total		100

Date: October 24, 2018.

Fill In the Blank

1 (5 Points). Assume that f is a function such that $f'(x)$ and $f''(x)$ are defined for all x.

(a) A point c is a *critical point* of f if

$f'(c) = 0$ or is undefined.

(b) f is *increasing* on an interval if $0 < f'(x)$ on that interval.

(c) f is *decreasing* on an interval if $f'(x) < 0$ on that interval.

(d) f is *concave up* on an interval if $0 < f''(x)$ on that interval.

(e) f is *concave down* on an interval if $f''(x) < 0$ on that interval.

2 (4 Points). The first derivative test says that a critical point, c, of f is:

a *local maximum* if f' changes from positive to negative at c;

local minimum if f' changes from negative to positive at c.

3 (2 Points). The second derivative test says that a critical point, c, of f is:

a *local maximum* if f'' is negative at c;

a *local minimum* if f'' is positive at c.

4 (1 Point). Suppose that $f''(c) = 0$. We say c is an inflection point of f if

f changes concavity at c.

5 (Mean Value Theorem - 3 Points). Let f be a function satisfying

1. f is Continuous on $[a, b]$, and
2. f is differentiable on (a, b).

Then there exists a c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

.

Problems

6 (15 Points)**.** Find dy/dx:

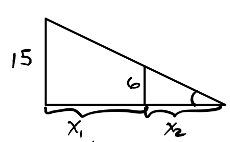
$$\cos(xy) = 1 + \tan y$$

$$\sec^2(y)\frac{dy}{dx} = -\sin(xy)\left[y + x\frac{dy}{dx}\right] = -y\sin(xy) - x\sin(xy)\frac{dy}{dx}$$

$$\Rightarrow \sec^2(y)\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx} = \frac{dy}{dx}\left[\sec^2(y) + x\sin(xy)\right] = -y\sin(xy)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-y\sin(xy)}{\sec^2(y) + x\sin(xy)}}$$

7 (20 Points)**.** A street light is mounted at the top of a 15-ft-tall pole. A person 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of their shadow moving when they are 40 ft from the pole?
(Hint: The length of the shadow is measured from the person to the tip of the shadow; the rate at which the tip of the shadow is moving is measured from the pole to the tip of the shadow.)



$x = x_1 + x_2$

We are given $\frac{dx_1}{dt} = 5$ and we want to find $\frac{dx}{dt}$. By similar triangles we have

$$\frac{6}{x_2} = \frac{15}{x} \Rightarrow 6x = 15x_2 = 15(x - x_1) = 15x - 15x_1$$

$$\Rightarrow 6x - 15x = -9x = -15x_1$$

$$\Rightarrow x = \frac{15}{9}x_1 = \frac{5}{3}x_1$$

Differentiating both sides yields

$$\frac{dx}{dt} = \frac{5}{3}\frac{dx_1}{dt} = \frac{5}{3}(5) = \boxed{\frac{25}{3}}$$

8 (15 Points). Find the value or values of c that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = x^3 - 2x^2 - 4x + 2$ on $[-2, 2]$.

$$f(2) = 8 - 2(4) - 4(2) + 2 = 8 - 8 - 8 + 2 = -6$$

$$f(-2) = -8 - 2(4) - 4(-2) + 2 = -8 - 8 + 8 + 2 = -6$$

$$\Rightarrow \frac{f(2) - f(-2)}{2 - (-2)} = \frac{-6 - (-6)}{4} = \frac{0}{4} = 0$$

$$0 = f'(x) = 3x^2 - 4x - 4 \iff x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)} = \frac{4 \pm \sqrt{16 + 16(3)}}{6}$$

$$= \frac{4 \pm \sqrt{4(16)}}{6} = \frac{4 \pm \sqrt{4}\sqrt{16}}{6} = \frac{4 \pm 2(4)}{6} = \frac{4(1 \pm 2)}{6} = \frac{2}{3}(1 \pm 2).$$

Either $c = \frac{2}{3}(1+2) = 2$, which is not in $(-2, 2)$, or $\boxed{c = \frac{2}{3}(1-2) = -\frac{2}{3}}$

9 (15 Points). Find the absolute maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$.

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1) = 0 \iff x = -1 \text{ or } x = 2.$$

$$f(-2) = 2(-8) - 3(4) - 12(-2) + 1 = -16 - 12 + 24 + 1 = -16 + 12 + 1 = -3.$$

$$f(-1) = 2(-1) - 3(1) - 12(-1) + 1 = -2 - 3 + 12 + 1 = 8.$$

$$f(2) = 2(8) - 3(4) - 12(2) + 1 = 16 - 12 - 24 + 1 = 4 - 24 + 1 = -20 + 1 = -19.$$

$$f(3) = 2(27) - 3(9) - 12(3) + 1 = 27(2-1) - 36 + 1 = 27 - 35 = -8.$$

$$\boxed{\begin{array}{l}\text{Absolute max: } (-1, 8) \\ \text{Absolute min: } (2, -19)\end{array}}$$

10 (20 Points)**.** Sketch the curve

$$f(x) = \frac{x^2}{x-1}$$

(a) State the domain of f.

$\boxed{x \neq 1}$

(b) Find the intercepts and express them as an (x, y) pair. Write NONE if there are none.

x-intercepts: $(0,0)$

y-intercept: $(0,0)$

(c) Is the function even, odd, or neither? What type of symmetry does the function have?

$f(-x) = \frac{(-x)^2}{-x-1} = \frac{x^2}{-(x+1)} \neq f(x), -f(x)$; $\boxed{\text{neither, no symmetry.}}$

(d) Find the asymptotes. Write NONE if there are none.

Horizontal: NONE

Oblique: $y = x+1$

Vertical: $x=1$

$\lim_{x\to\infty} \frac{x^2}{x-1} = \infty$, $\lim_{x\to-\infty} \frac{x^2}{x-1} = -\infty$

$\lim_{x\to 1^+} \frac{x^2}{x-1} = \infty$, $\lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \\ & -x^2+x \\ \hline & x \\ & -x+1 \\ \hline & 1 \end{array}$$

$\frac{x^2}{x-1} = x+1 + \frac{1}{x-1}$

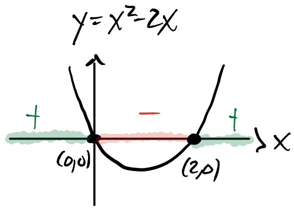
$$\lim_{x\to\pm\infty}\left(\frac{x^2}{x-1} - (x+1)\right) = \lim_{x\to\pm\infty}\left(\frac{x^2-(x+1)(x-1)}{x-1}\right)$$
$$= \lim_{x\to\pm\infty}\left(\frac{x^2-(x^2-1)}{x-1}\right)$$
$$= \lim_{x\to\pm\infty} \frac{1}{x-1} = 0.$$

(e) Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(-\infty, 0)$ and $(2, \infty)$

Decreasing: $(0,1) \cup (1,2)$ (since $x=1$ is not in the domain of f)

$$f'(x) = \frac{2x(x-1) - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$



(f) State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(0,0)$

Local minimum value(s): $(2,4)$

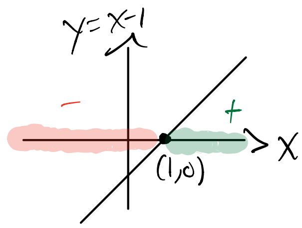
(g) Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

Concave Up: $(1, \infty)$

Concave Down: $(-\infty, 1)$

Inflection Points: NONE

$$f''(x) = \frac{(2x-2)(x-1)^2 - (x^2-2x)(2)(x-1)(1)}{(x-1)^2}$$



$$\frac{2(x-1)\left[(x-1)^2 - x(x-2)\right]}{(x-1)^4} = \frac{2\left[\cancel{x^2} - \cancel{2x} + 1 - \cancel{x^2} + \cancel{2x}\right]}{(x-1)^3}$$

$$= \frac{2}{(x-1)^3} \neq 0$$

(h) Use your answers to Parts (a)-(g) to sketch the curve. Be sure that your graph is labeled and neat. Messy/incoherent graphs will receive zero points.