EXAM 2

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Date: October 23, 2019.

Functions

1. Find the value of m that makes the function

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x+5} & \text{if } -5 < x \\ mx + 15 & \text{if } x \leq -5 \end{cases}$$

a continuous function.

$$\lim_{x \to -5^-} f(x) = \lim_{x \to -5^-} \frac{(x+5)(x-5)}{x+5}$$

$$= \lim_{x \to -5^-} x-5 = -10$$

$$\lim_{x \to -5^+} f(x) = \lim_{x \to -5^+} mx+15 = -5m+15 = f(-5)$$

$$-5m+15 = -10$$

$$\Rightarrow -5m = -25$$

$$\Rightarrow \boxed{m = 5}$$

Derivatives

2. Use the **limit definition**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \frac{1}{x}.$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \boxed{\frac{-1}{x^2}}$$

Derivative Rules

3. Compute the line tangent to

$$f(x) = \tan(x)$$

at $x = \pi/4$.

$$f(\pi/4) = \tan(\pi/4) = 1$$

$$f'(x) = \sec^2(x)$$

$$f'(\pi/4) = \sec^2(\pi/4)$$

$$= \frac{1}{\cos^2(\pi/4)}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= 2$$

$$\boxed{y - 1 = 2(x - \pi/4)}$$

Product and Quotient Rules

4. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\sec(x)\right].$$

$$\boxed{2x\sec(x) + x^2\sec(x)\tan(x)}$$

5. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{x+1}\right].$$

$$\boxed{\frac{-\sin(x)(x+1) - \cos(x)}{(x+1)^2}}$$

Chain Rule

6. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos\left(\frac{x+1}{x+2}\right)\right].$$

$$-\sin\left(\frac{x+1}{x+2}\right)\left(\frac{x+2-(x+1)}{(x+2)^2}\right)$$

$$= -\sin\left(\frac{x+1}{x+2}\right)\left(\frac{1}{(x+2)^2}\right)$$

$$= \boxed{\frac{-\sin\left(\frac{x+1}{x+2}\right)}{(x+2)^2}}$$

Implicit Differentiation and Related Rates

7. The volume of a tree is given by $V = \frac{1}{12\pi}C^2h$, where C is the circumference of the tree in meters at ground level, and h is the height of the tree in meters. Both C and h are functions of time, t, in years.

(a) Find a formula for $\frac{dV}{dt}$. What does $\frac{dV}{dt}$ represent in practical terms?

$V' = \frac{1}{12\pi}\left(2CC'h + C^2h'\right)$ represents the rate of change of the volume of a tree with respect to time.

(b) Suppose the circumference grows at a rate of 1 meter every 5 years, and the height grows at a rate of 4 meters per year. How fast is the volume of the tree growing when the circumference is 5 meters and the height is 22 meters?

$C' = 1/5,\ h' = 4,\ C = 5,\ h = 22$

$$V' = \frac{1}{12\pi}\left(2(5)\left(\tfrac{1}{5}\right)(22) + 5^2(4)\right)$$

$$= \frac{1}{12\pi}\left(44 + 100\right)$$

$$= \frac{144}{12\pi}$$

$$= \boxed{\frac{12}{\pi} \text{ meters}^3/\text{year}}$$

Derivatives and Shape

Use the function

$$f(x) = x^3 - 6x^2 + 9x - 4$$

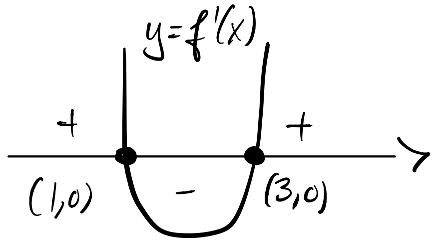
to answer the following questions.

8. Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(-\infty, 1) \cup (3, \infty)$

Decreasing: $(1, 3)$

$$f'(x) = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-1)(x-3)$$



$$f(1) = 1 - 6 + 9 - 4 = 0$$
$$f(3) = 27 - 54 + 27 - 4 = -4$$

9. State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(1, 0)$

Local minimum value(s): $(3, -4)$

10. Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

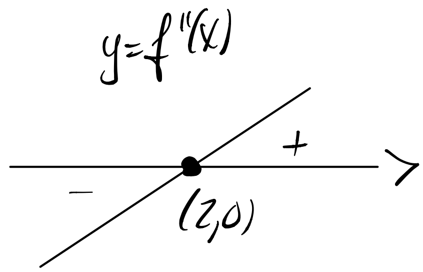
Concave Up: $(2,\infty)$

Concave Down: $(-\infty,2)$

Inflection Points: $(2,-2)$

$$f''(x) = 6x - 12 = 6(x-2)$$

$$f(2) = 8-24+18-4$$
$$= 26-28$$
$$= -2$$



Asymptotes

11. Find the asymptotes of

$$f(x) = \frac{3x^2}{x^2 - 4}$$

Write NONE if there are none.

Horizontal: $y = 3$

Vertical: $x = \pm 2$

$$\lim_{x \to \infty} \frac{3x^2}{x^2-4} = \lim_{x \to \infty} \frac{x^2}{x^2}\left(\frac{3}{1 - 4/x^2}\right)$$

$$= \lim_{x \to \infty} \frac{3}{1 - 4/x^2}$$

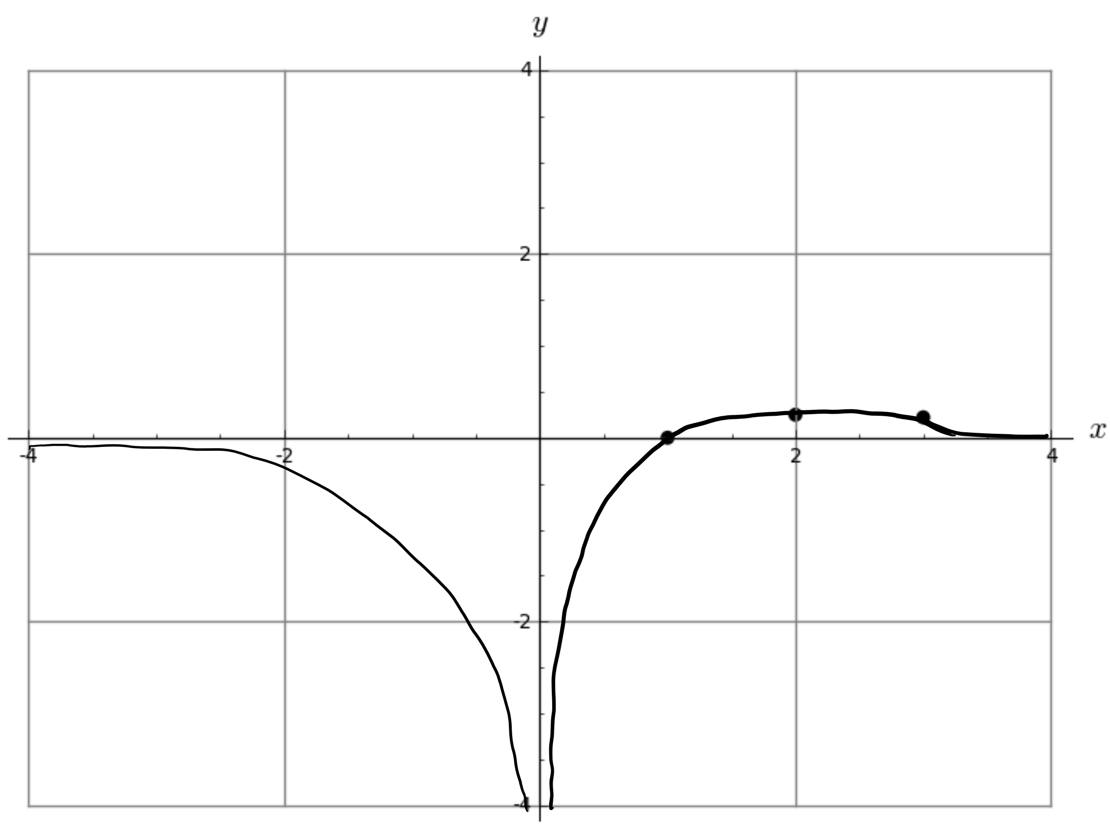
$$= \frac{3}{1-0} = 3$$

Curve Sketching

12. You are given the following information about a function, f:

- x-intercept(s): $(1, 0)$,
- y-intercept: None,
- asymptotes: $x = 0$ and $y = 0$,
- critical point(s): $(2, 1/4)$,
- increasing: $(0, 2)$,
- decreasing: $(-\infty, 0) \cup (2, \infty)$,
- concave up: $(3, \infty)$,
- concave down: $(-\infty, 3)$, and
- $f(3) = 2/9$.

From left to right, the points $(1, 0)$, $(2, 1/4)$, and $(3, 2/9)$ have been plotted below. Use the information above to sketch the curve.



Closed Interval Method and Optimization

13. Find the absolute maximum and minimum values of $f(x) = x^2 - 4x - 12$ on $[0, 5]$.

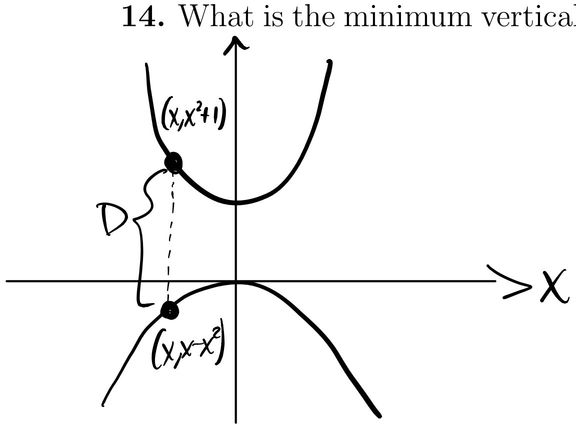
$f'(x) = 2x - 4 = 2(x-2) = 0 \Rightarrow x = 2$

$f(0) = -12$

$f(2) = 4 - 8 - 12 = \boxed{-16}$ min

$f(5) = 25 - 20 - 12 = \boxed{-7}$ max

14. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?



$D = x^2 + 1 - (x - x^2)$

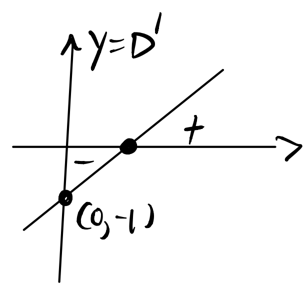
$= x^2 + 1 - x + x^2$

$= 2x^2 - x + 1$

$D' = 4x - 1 = 0$

$\Rightarrow x = 1/4$

is a global minimum



so

$D(1/4) = 2\left(\frac{1}{16}\right) - \frac{1}{4} + 1$

$= \frac{1}{8} - \frac{2}{8} + \frac{8}{8}$

$= \boxed{\frac{7}{8}}$