# EXAM 2

BLAKE FARMAN

*Lafayette College*

Answer the questions below and submit them through Moodle before time expires.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**

Name: Solutions

*Date*: April 5, 2020.

## Functions

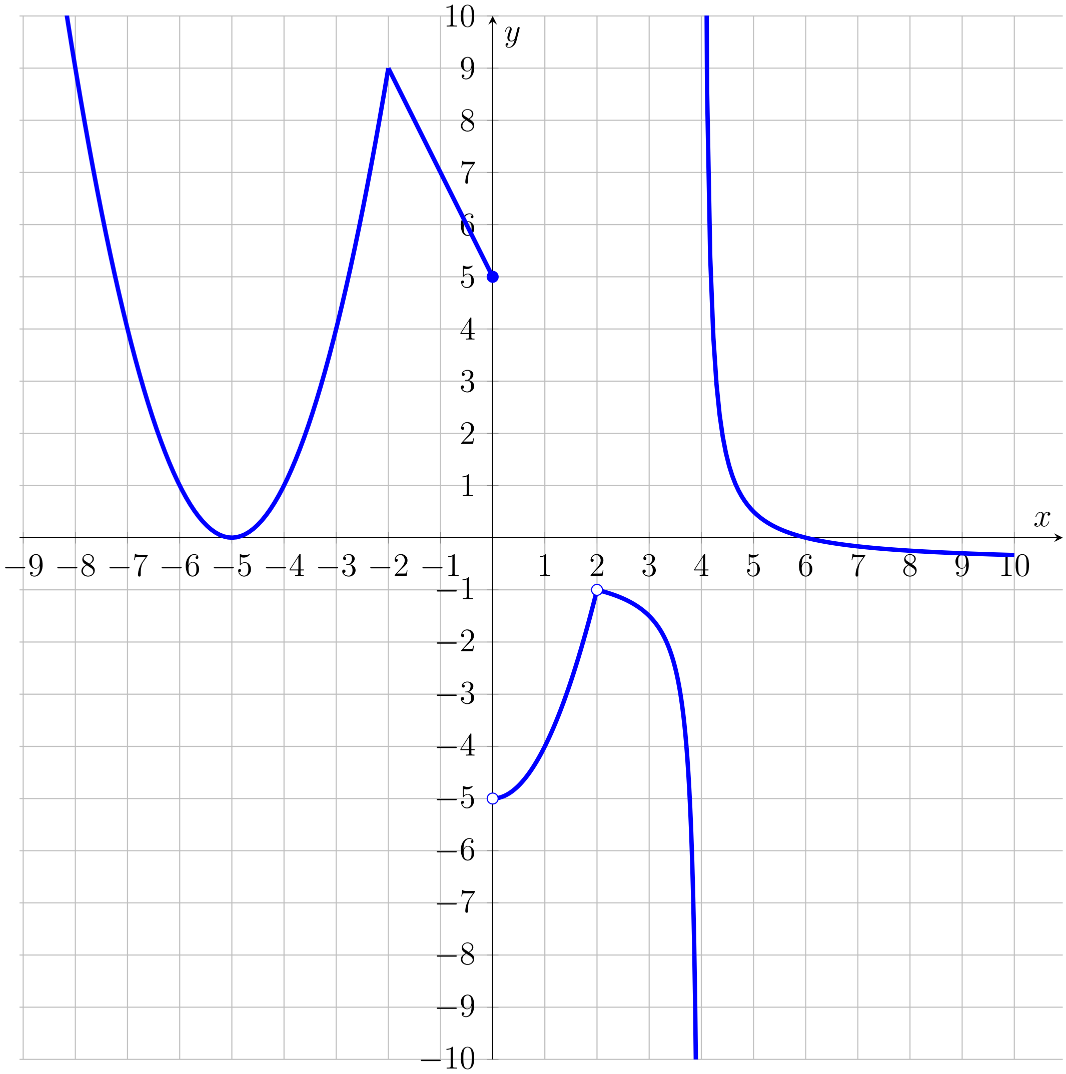


Figure 1. The graph of $f$.

**1.** Use Figure 1 to answer the following questions.

a. State all values $a$ for which $\lim_{x \to a} f(x)$ does not exist. Justify your answers.

$a = 0$

$\lim_{x \to 0^-} f(x) = 5$

$\lim_{x \to 0^+} f(x) = -5$

$a = 4$

$\lim_{x \to 4^-} f(x) = -\infty$

$\lim_{x \to 4^+} f(x) = \infty$

b. State all values $a$ for which $f$ is discontinuous. Use the definition of continuity to justify your answers.

$a = 0$

$\lim_{x \to 0} f(x)$ does not exist

$a = 4$

$\lim_{x \to 4} f(x)$ does not exist

$a = 2$

$\lim_{x \to 2} f(x) = -1$ but

$f(-1)$ is undefined

## Derivatives

**1.** Use the **limit definition**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \sqrt{x}.$$

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

## Derivative Rules

**1.** Consider the following differentiable functions, $f$ and $g$, satisfying

- $f(2) = 1$ and $f'(2) = 3$,
- $g(2) = 3$ and $g'(2) = 1$.

Compute the line tangent to the function

$$h(x) = f(x) + x^2 - g(x)$$

at $x = 2$.

$$h(2) = f(2) + 2^2 - g(2) = 1 + 4 - 3 = 2$$

$$h'(x) = f'(x) + 2x - g'(x)$$

$$h'(2) = f'(2) + 4 - g'(2) = 3 + 4 - 1 = 6$$

$\boxed{y - 2 = 6(x-2)}$ or $\boxed{y = 6x - 10}$

## Product and Quotient Rules

Assume that $f$ is a differentiable function satisfying

$$f\left(\frac{\pi}{2}\right) = 2 \quad \text{and} \quad f'\left(\frac{\pi}{2}\right) = 4.$$

Use this function to compute $g'(\pi/2)$ and $h'(\pi/2)$ below.

**1.** $g(x) = \sin(x)f(x)$

$$g'(x) = \cos(x)f(x) + \sin(x)f'(x)$$

$$g'\left(\tfrac{\pi}{2}\right) = \cos(\pi/2)f\left(\tfrac{\pi}{2}\right) + \sin\left(\tfrac{\pi}{2}\right)f'\left(\tfrac{\pi}{2}\right)$$

$$= 0 + f'\left(\tfrac{\pi}{2}\right)$$

$$= \boxed{4}$$

**2.** $h(x) = \dfrac{\sin(x)}{f(x)}$.

$$h'(x) = \frac{\cos(x)f(x) - \sin(x)f'(x)}{f(x)^2}$$

$$h'\left(\tfrac{\pi}{2}\right) = \frac{0-4}{2^2} = \boxed{-1}$$

## Chain Rule

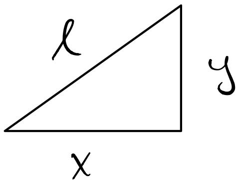
**1.** Assume that $f$ is a differentiable function satisfying $f'(9) = 4$ and let $g(x) = f(x^2)$. Compute $g'(3)$.

$$g'(x) = f'(x^2)\frac{d}{dx}x^2 = 2x\,f'(x^2)$$

$$g'(3) = 2(3)\,f'(3^2) = 6(4) = \boxed{24}$$

## Implicit Differentiation and Related Rates

**1.** The top of a ladder slides down a vertical wall at a rate of 3 meters/second. At the moment when the top of the ladder is 4 meters from the ground, it slides away from the wall at a rate of 4 meters/second. How long is the ladder?




Given: $y' = \frac{dy}{dt} = -3\,m/s$

When $y = 4m$, $x' = \frac{dx}{dt} = 4\,m/s$

Want: $\ell$

Need: $x$ when $y = 4m$.

$$\ell^2 = x^2 + y^2 \iff \ell = \sqrt{x^2 + y^2}$$

$$\Rightarrow 2\ell\ell' = 0 = 2xx' + 2yy'$$

$$\Rightarrow xx' = -yy'$$

$$\Rightarrow x = \frac{-yy'}{x'} = \frac{-4(-3)}{4} = 3m$$

Therefore

$$\ell = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = \boxed{5m}$$

## Derivatives and Shape

Use the function

$$f(x) = x^3 - 6x^2 + 9x - 4$$

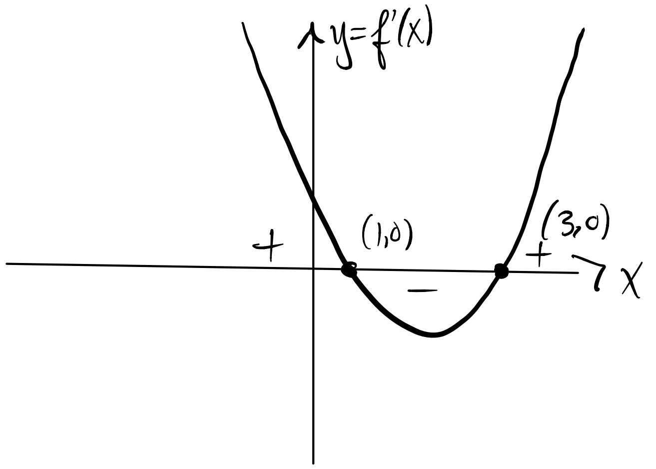
to answer the following questions.

**1.** Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(-\infty, 1) \cup (3, \infty)$

Decreasing: $(1, 3)$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$




**2.** State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(1, 0)$

Local minimum value(s): $(3, -4)$

$f(x) = x^3 - 6x^2 + 9x - 4$

$f(1) = 1^3 - 6(1)^2 + 9(1) - 4 = 1 - 6 + 9 - 4 = 10 - 10 = 0$

$f(3) = 3^3 - 6(3)^2 + 9(3) - 4$

$= 27 - 54 + 27 - 4$

$= -4$

**3.** Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

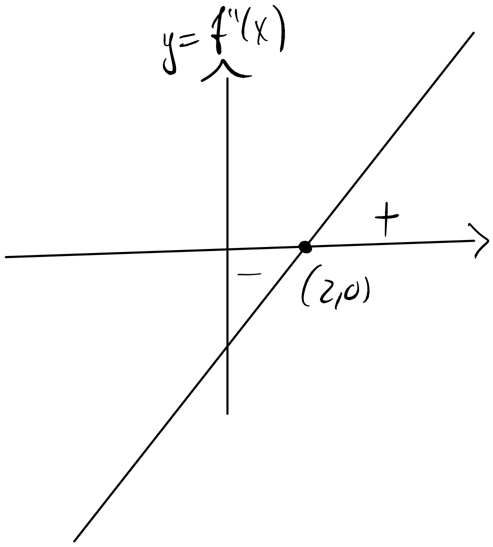
Concave Up: $(2, \infty)$

Concave Down: $(-\infty, 2)$

Inflection Points: $(2, -2)$

$$f''(x) = 6x - 12 = 6(x-2)$$

$$f(2) = 2^3 - 6(2)^2 + 9(2) - 4$$
$$= 8 - 24 + 18 - 4$$
$$= 4 - 6$$
$$= -2$$



## Asymptotes

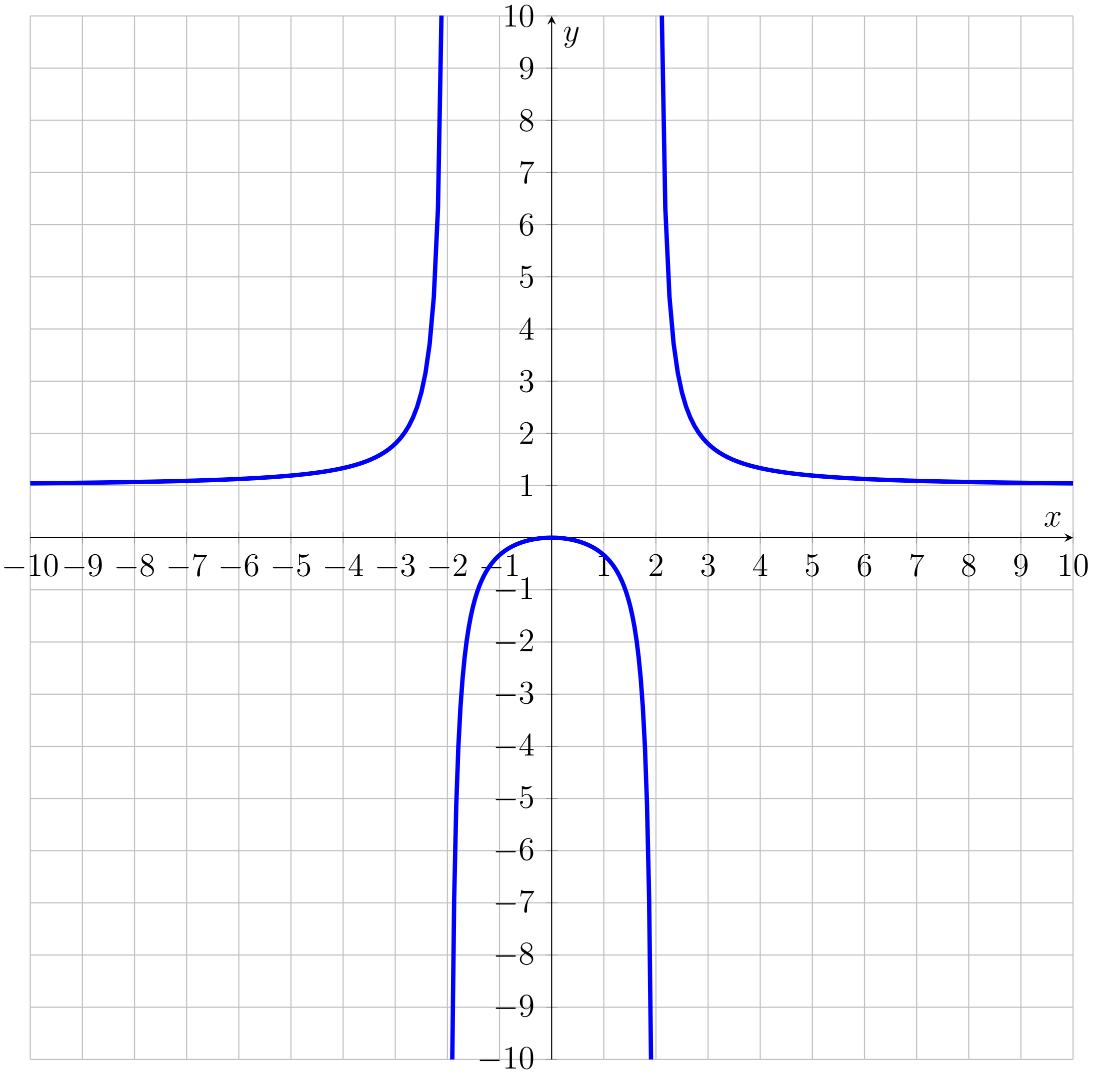


Figure 2. The graph of $f$

Use Figure 2 to answer the following questions.

**1.** List any vertical asymptotes of the function $f$. Justify your answers using limits.

$X = 2$

$\lim_{X \to 2^-} f(X) = -\infty$

$\lim_{X \to 2^+} f(X) = \infty$

$X = -2$

$\lim_{X \to -2^+} f(X) = -\infty$

$\lim_{X \to -2^-} f(X) = \infty$

**2.** List any horizontal asymptotes of the function $f$. Justify your answers using limits.

$y = 1$

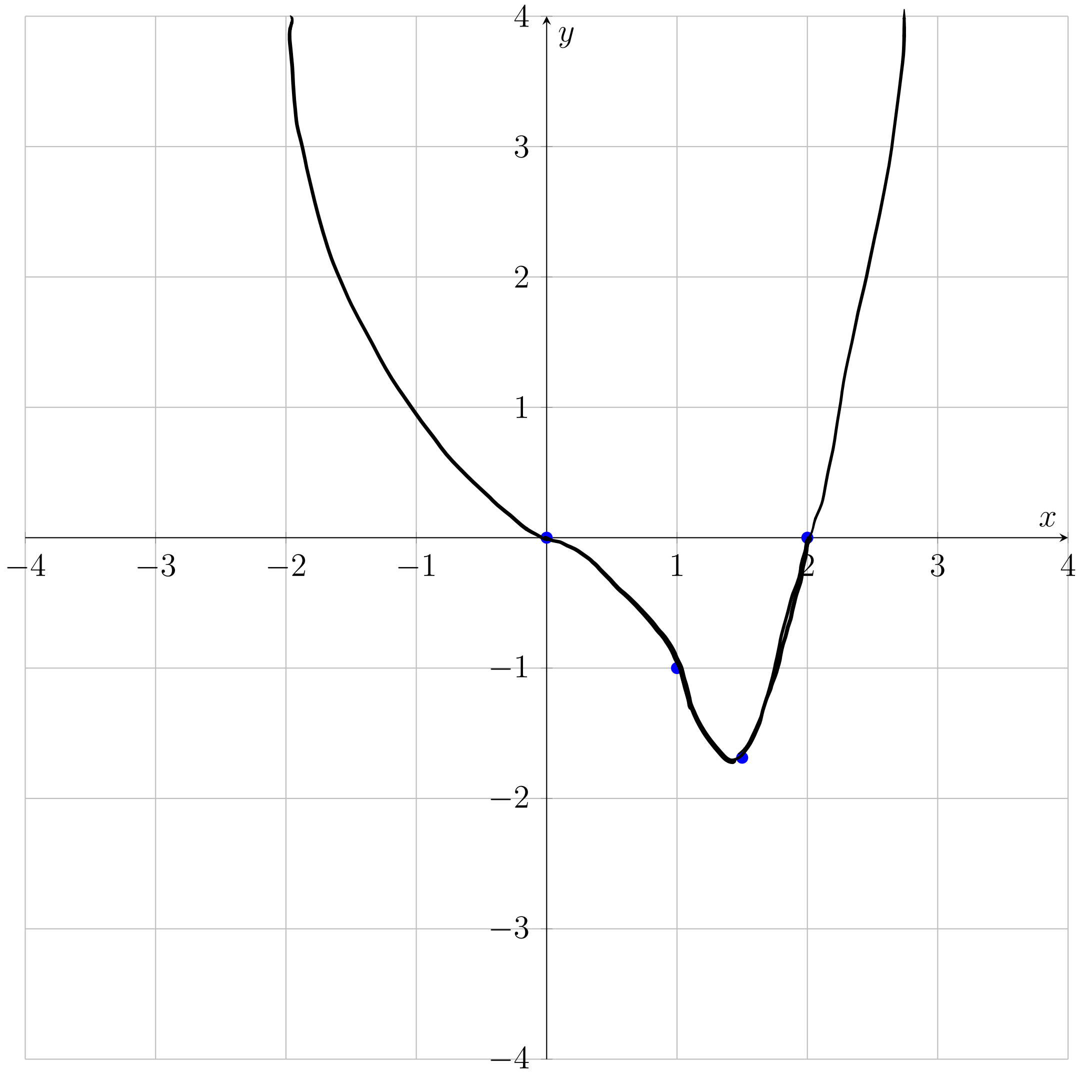
$\lim_{X \to \infty} f(X) = \lim_{X \to -\infty} f(X) = 1.$

## Curve Sketching

**1.** You are given the following information about a function, $f$:

- $x$-intercepts: $(0,0)$ and $(2,0)$,
- $y$-intercepts: $(0,0)$,
- asymptotes: none,
- critical points: $(0,0)$ and $\left(\frac{3}{2}, -\frac{27}{16}\right)$,
- increasing: $\left(\frac{3}{2}, \infty\right)$,
- decreasing: $(-\infty, 3/2)$,
- concave up: $(-\infty, 0) \cup (1, \infty)$,
- concave down: $(0, 1)$, and
- $f(1) = -1$.

Use this information to sketch the curve. From left to right, the points $(0,0)$, $(1,-1)$, $(3/2, -27/16)$, and $(2,0)$ have been plotted for you.



## Closed Interval Method and Optimization

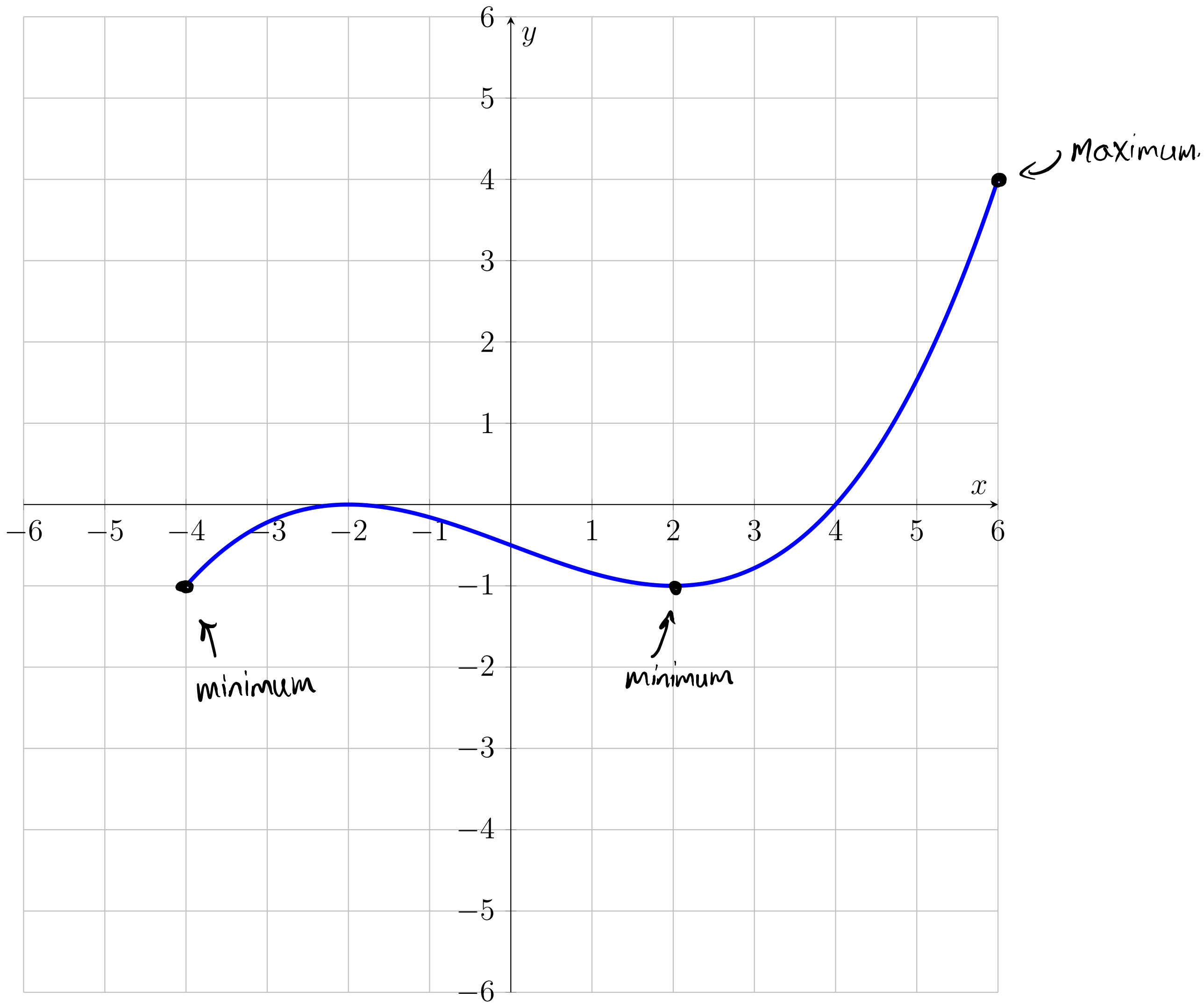


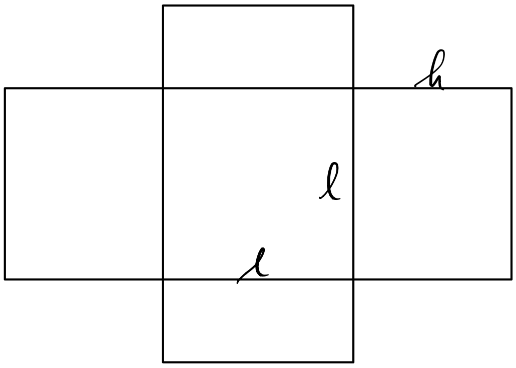

Figure 3. The graph of $f$

**1.** Use Figure 3 to find the absolute maximum and minimum values of $f$ on the interval $[-4, 6]$. List your solutions as an $(x, y)$ pair.

Absolute Minima: $(-4, -1)$ and $(2, -1)$

Absolute Maximum: $(6, 4)$

**2.** A box with a square base and no top must have a volume of $4\,\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.



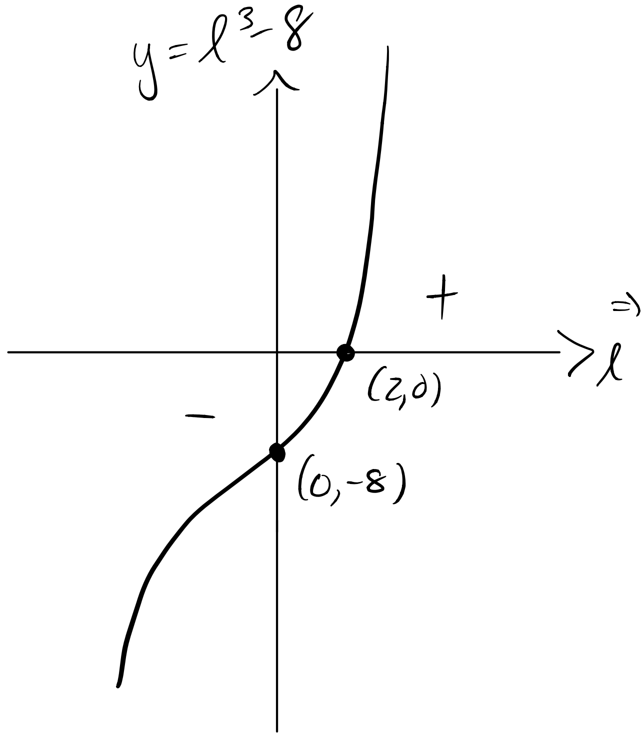

$$V=\ell^2 h=4 \Rightarrow h=\frac{4}{\ell^2}$$

$$S=4\ell h+\ell^2=\ell^2+\frac{16}{\ell}$$

$$S'=2\ell-\frac{16}{\ell^2}=\frac{2\ell^3-16}{\ell^2}=\frac{2}{\ell^2}(\ell^3-8)=0$$

$$\Rightarrow \ell^3=8$$

$$\Rightarrow \boxed{\ell=2\text{ cm},\ h=\frac{4}{2^2}=1\text{ cm}}$$




$\Rightarrow S'(\ell)>0$ on $(2,\infty)$

$S'(\ell)<0$ on $(-\infty,2)$

$\Rightarrow \ell=2$ corresponds to an absolute minimum by the First Derivative Test.