EXAM 2

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Date: October 11, 2019.

Inverse Trig Functions

1. Compute

$$\int \frac{\sec^2(x)}{1+\tan^2(x)}\,\mathrm{d}x$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$\int \frac{\sec^2(x)}{1+\tan^2(x)}\,dx = \int \frac{1}{1+u^2}\,du$$

$$= \arctan(u) + C$$

$$= \arctan(\tan(x)) + C$$

$$= \boxed{x + C}$$

L'Hôpital's Rule

2. Compute
$$\lim_{x\to\infty}\left[\ln(x^2-4)-\ln(x+2)\right]$$

$$\lim_{x\to\infty}\left[\ln(x^2-4)-\ln(x+2)\right]=\lim_{x\to\infty}\ln\left(\frac{x^2-4}{x+2}\right)$$

Since
$$\lim_{x\to\infty}\frac{x^2-4}{x+2}\overset{L'H}{=}\lim_{x\to\infty}\frac{2x}{1}=\lim_{x\to\infty}2x=\infty$$

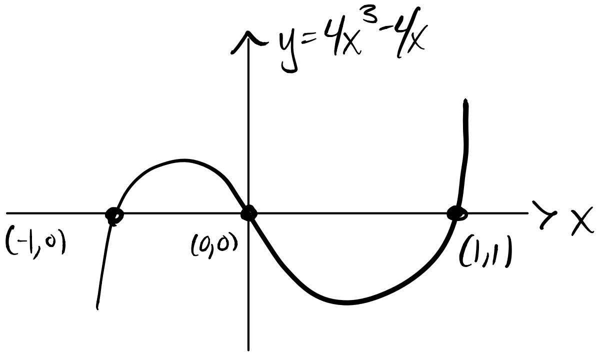
we can let $y=\frac{x^2-4}{x+2}$ and rewrite the original limit

$$\lim_{x\to\infty}\ln\left(\frac{x^2-4}{x+2}\right)=\lim_{y\to\infty}\ln(y)=\boxed{\infty}$$

Area and Volumes

3. Compute the area of the region bounded by $f(x) = 4x^3 - x$ and $g(x) = 3x$.

$$f(x) - g(x) = 4x^3 - x - 3x = 4x^3 - 4x = 4x(x^2-1) = 4x(x+1)(x-1)$$



$$A = \int_{-1}^{0} 4x^3 - 4x\,dx + \int_0^1 4x - 4x^3\,dx$$

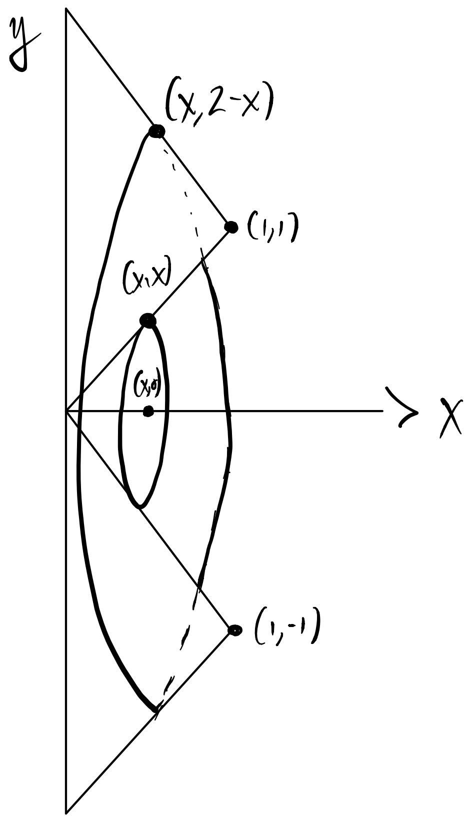
$$= x^4\Big|_{-1}^{0} - 2x^2\Big|_{-1}^{0} + 2x^2\Big|_0^1 - x^4\Big|_0^1$$

$$= (0-1) - 2(0-1) + 2(1-0) - (1-0)$$

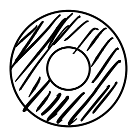
$$= -1 + 2 + 2 - 1$$

$$= \boxed{2}$$

4. Compute the volume of the solid obtained by revolving the region bounded by the three lines $x = 0$, $y = 2 - x$, and $y = x$ about the x-axis.



Cross-Section



$$A(x) = \pi(2-x)^2 - \pi x^2$$

$$= \pi\left[4 - 4x + x^2 - x^2\right]$$

$$= 4\pi(1-x)$$

$$V = \int_0^1 A(x)\,dx = 4\pi\int_0^1 (1-x)\,dx$$

$$= 4\pi\left(x\Big|_0^1 - \tfrac{1}{2}x^2\Big|_0^1\right) = 4\pi(1 - 1/2)$$

$$= \boxed{2\pi}$$

Integration by Parts

5. Compute

$$\int x^2 e^x \, \mathrm{d}x$$

$u = x^2 \qquad v = e^x$

$du = 2x \qquad dv = e^x dx$

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx$$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$$= x^2 e^x - 2\left(x e^x - \int e^x dx\right)$$

$$= \boxed{x^2 e^x - 2xe^x + 2e^x + C}$$

Partial Fractions

6. Compute
$$\int \frac{x+1}{(2x+3)(3x+5)}\,dx$$

$$\frac{x+1}{(2x+3)(3x+5)} = \frac{A}{2x+3} + \frac{B}{3x+5} \Rightarrow x+1 = A(3x+5) + B(2x+3)$$

$\underline{x = -\frac{5}{3}}$

$$-\frac{5}{3} + \frac{3}{3} = -\frac{2}{3} = A(0) + B\left(-\frac{10}{3} + \frac{9}{3}\right) = -\frac{B}{3} \Rightarrow B = 2$$

$\underline{x = -\frac{3}{2}}$

$$-\frac{3}{2} + \frac{2}{2} = -\frac{1}{2} = A\left(-\frac{9}{2} + \frac{10}{2}\right) + B(0) = \frac{A}{2} \Rightarrow A = -1.$$

$$\int \frac{x+1}{(2x+3)(3x+5)}\,dx = \int \frac{-1}{2x+3}\,dx + \int \frac{2}{3x+5}\,dx = \boxed{-\frac{1}{2}\ln|2x+3| + \frac{2}{3}\ln|3x+5| + C}$$

<u>Check:</u> $$\frac{d}{dx}\left(-\frac{1}{2}\ln|2x+3| + \frac{2}{3}\ln|3x+5| + C\right) = -\frac{1}{2}\left(\frac{2}{2x+3}\right) + \frac{2}{3}\left(\frac{3}{3x+5}\right)$$

$$= \frac{-1}{2x+3} + \frac{2}{3x+5} = \frac{-3x-5+4x+6}{(2x+3)(3x+5)}$$

$$= \frac{x+1}{(2x+3)(3x+5)}$$

Approximate Integration

Use the function $f(x) = 2x^3 - x + 1$ to answer Problems 7 and 8.

7. Use the inequality

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ on $[a,b]$, to find the number of intervals needed to estimate

$$\int_0^4 f(x)\,\mathrm{d}x$$

using the Trapezoidal Rule with an error less than 10^{-2}.

$f'(x) = 6x^2 - 1$

$f''(x) = 12x$

The range of $12x$ on $[0,4]$ is $[0,48]$, so

$$|f''(x)| \leq 48$$

implies

$$|E_T| \leq \frac{48(4-0)^3}{12n^2} = \frac{4(4)^3}{n^2} < \frac{1}{10^2}$$

So

$$4^4 10^2 = (4^2 10)^2 = 160^2 < n^2$$

Therefore we need $\boxed{\text{at least 161 intervals}}$

8. Use the Trapezoid Rule

$$T_n = \frac{f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)}{2}\Delta x$$

with $n = 2$ intervals to estimate the value of the integral

$$\int_0^4 f(x)\,\mathrm{d}x.$$

$f(x) = 2x^3 - x + 1$, $\Delta x = \frac{4-0}{2} = 2$, $x_0 = 0$, $x_1 = 2$, $x_2 = 4$

$f(x_0) = f(0) = 1$

$f(x_1) = f(2) = 2(8) - 2 + 1 = 16 - 1 = 15$

$f(x_2) = f(4) = 2(64) - 4 + 1 = 128 - 3 = 125$

$T_2 = \frac{1 + 2(15) + 125}{2} \cdot 2 = 1 + 30 + 125 = \boxed{156}$

Improper Integrals

9. Compute the improper integral

$$\int_1^\infty \frac{1}{x^2+1}\,\mathrm{d}x$$

$$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx$$

$$= \lim_{t\to\infty} \arctan(x)\Big|_1^t$$

$$= \lim_{t\to\infty}\left(\arctan(t) - \arctan(1)\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \boxed{\frac{\pi}{4}}$$

Differential Equations

10. Solve the inital value problem

$$y' = \frac{2x}{3y^2}, \quad y(2) = 2.$$

$$\int 3y^2\,dy = \int 2x\,dx$$

$$\Rightarrow \quad y^3 = x^2 + C$$

$$\Rightarrow \quad 8 = 4 + C \Rightarrow C = 4$$

$$\boxed{y = \sqrt[3]{x^2 + 4}}$$

Parametric Curves

11. Eliminate the parameter to find a Cartesian equation for the curve

$$x = t + 3, \quad y = t^2 + 6t + 8, \quad -4 \le t \le -2$$

then sketch the curve and indicate with an arrow the direction in which the curve is traced as t increases.

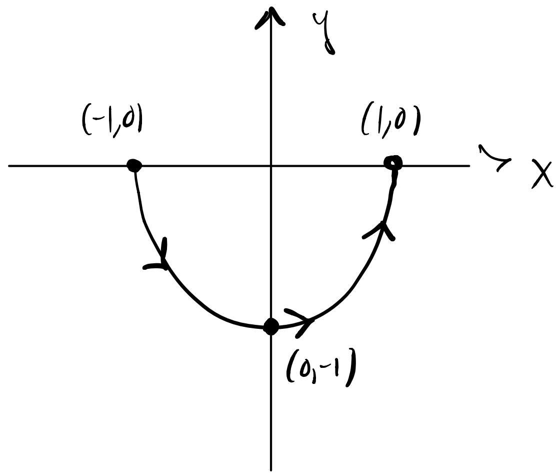
$$\begin{aligned} t^2+6t+8 &= (t+4)(t+2) \\ &= (t+3+1)(t+3-1) \\ &= (x+1)(x-1) = x^2-1 \end{aligned}$$

$\underline{t=-4}$

$x = -4+3 = -1, \quad y = 0$

$\underline{t=-2}$

$x = -2+3 = 1, \quad y = 0$



Polar Coordinates

12. Sketch the rose $r = \sin(2\theta)$, $0 \le \theta \le 2\pi$. Label the tips of the petals and draw arrows to indicate the direction in which the curve is traced.