EXAM 2

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
By writing your name on the line below, you indicate that you have read and understand these directions.

Name: Solutions

Problem	Points Earned	Points Possible
1		15
2		10
3		12
4		13
5		25
6		25
Total		100

Date: March 27, 2019.

Approximate Integration

Use the function $f(x) = 2x^3 - x + 1$ to answer Problems 1 and 2.

1. Use the inequality

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ on $[a, b]$, to find the number of intervals needed to estimate

$$\int_0^4 f(x)\,\mathrm{d}x$$

using the Trapezoidal Rule with an error less than 10^{-2}.

$f'(x) = 6x^2 - 1$

$f''(x) = 12x$

$f'''(x) = 12 \neq 0$

By the Closed Interval Method, the absolute maximum/absolute minimum is the largest/smallest of

$$f(0) = 0,\ f(4) = 48.$$

The largest of these two values in absolute value is 48, so on $[0,4]$

$$-48 \leq f''(x) \leq 48 \iff |f''(x)| \leq 48$$

$$\Rightarrow |E_T| < \frac{48(4-0)^3}{12n^2} = \frac{4(4)^3}{n^2} = \frac{4^4}{n^2} \leq \frac{1}{10^2}$$

$$\Rightarrow 4^4 \cdot 10^2 \leq n^2$$

$$\Rightarrow \sqrt{4^4 10^2} = \sqrt{4^4}\sqrt{10^2} = (4^4)^{1/2}(10^2)^{1/2} = 4^{4/2} \cdot 10^{2/2} = 4^2 \cdot 10 = 16 \cdot 10 = 160 \leq \sqrt{n^2} = n$$

Therefore if we use at least 160 intervals the error will be smaller than $\frac{1}{100} = 0.01$.

2. Use the Trapezoid Rule

$$T_n = \frac{f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)}{2}\Delta x$$

with $n = 2$ intervals to estimate the value of the integral

$$\int_0^4 f(x)\,\mathrm{d}x.$$

$$T_2 = \frac{f(x_0)+2f(x_1)+f(x_2)}{2}\Delta x \qquad x_0 = a,\ x_1 = a+\Delta x,\ x_2 = a+2\Delta x = b,\ \Delta x = \frac{b-a}{n}$$

$f(x) = 2x^3 - x + 1$, $a = 0$, $b = 4$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{2} = \frac{4}{2} = 2$$

$x_0 = 0$, $x_1 = 0+2 = 2$, $x_2 = 0 + 2(2) = 4$.

$f(x_0) = f(0) = 2\cdot 0^3 - 0 + 1 = 1$

$f(x_1) = f(2) = 2(2^3) - 2 + 1 = 2(8) - 1 = 16 - 1 = 15$

$f(x_2) = f(4) = 2\cdot 4^3 - 4 + 1 = 2\cdot 4^3 - 3$ $\qquad 4^3 = 4\cdot 16 = 64$

$= 2(64) - 3 = 128 - 3 = 125$

$$T_2 = \frac{f(x_0)+2f(x_1)+f(x_2)}{2}\cdot 2 = f(x_0) + 2f(x_1) + f(x_2)$$

$$= 1 + 2(15) + 125 = 1 + 30 + 125 = \boxed{156}$$

Differential Equations

Solve the initial value problems.

3. $y' = 2yx$, $y(0) = \frac{1}{e}$

$dy/dx = 2yx$

$\Rightarrow \frac{1}{y}dy = 2x\,dx$

$\Rightarrow \int \frac{1}{y}dy = \int 2x\,dx$

$\Rightarrow \ln|y| = 2(\frac{1}{2}x^2) + C = x^2 + C$

$\Rightarrow |y| = e^{x^2+C} = e^C e^{x^2}$

$\Rightarrow y = \pm e^C e^{x^2}$

$\frac{1}{e} = \pm e^C e^{0^2} = \pm e^C(1) = \pm e^C$

$\Rightarrow \frac{1}{e} = e^C$

$\Rightarrow \boxed{y = \frac{1}{e}e^{x^2} = \frac{e^{x^2}}{e} = e^{x^2-1}}$

4. $y' = e^{-y}(2x - 4)$, $y(3) = 0$.

$dy/dx = e^{-y}(2x-4) = \frac{2x-4}{e^y}$

$\Rightarrow e^y dy = (2x-4)dx$

$\Rightarrow \int e^y dy = \int (2x-4)dx = 2\int x\,dx - 4\int dx$

$\Rightarrow e^y = 2(\frac{1}{2}x^2) - 4x + C = x^2 - 4x + C$

$\Rightarrow \ln(e^y) = \ln(x^2 - 4x + C)$

$\Rightarrow y = \ln(x^2 - 4x + C)$

$0 = \ln(9 - 12 + C) = \ln(-3 + C)$

$\Rightarrow e^0 = e^{\ln(-3+C)}$

$\Rightarrow 1 = -3 + C$

$\Rightarrow 1 + 3 = 3 - 3 + C$

$\Rightarrow 4 = C$

$$\boxed{\begin{aligned} y &= \ln(x^2 - 4x + 4) \\ &= \ln((x-2)^2) \\ &= 2\ln(x-2) \end{aligned}}$$

Parametric Equations

5. Eliminate the parameter to find a Cartesian equation for the curve

$$x = t + 2,\ y = t^2 + t - 6,\ -3 \leq t \leq 2$$

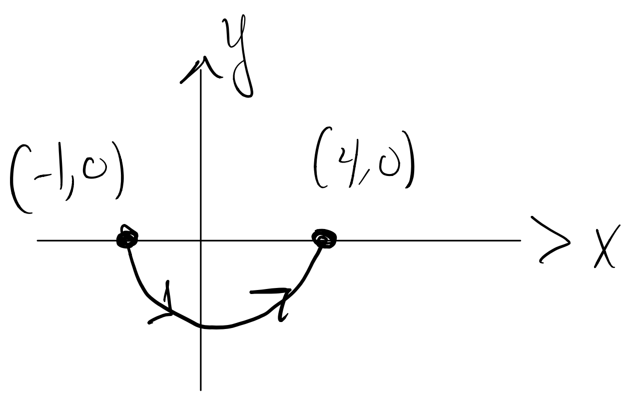
then sketch the curve and indicate with an arrow the direction in which the curve is traced as t increases.

$x = t + 2 \Rightarrow t = x - 2$

$y = t^2 + t - 6$
$= (t+3)(t-2)$
$= (x-2+3)(x-2-2)$
$= (x+1)(x-4)$
$= x^2 - 3x - 4.$

$t = x - 2 \Rightarrow -3 \leq x - 2 \leq 2$
$\Rightarrow -3+2 \leq x-2+2 \leq 2+2$
$\Rightarrow -1 \leq x \leq 4$

$y = (x+1)(x-4)$ on $[-1,4]$ is the part of the parabola between the roots



Polar Coordinates

6. Sketch the rose $r = \sin(2\theta)$, $0 \le \theta \le 2\pi$. Label the tips of the petals and draw arrows to indicate the direction in which the curve is traced.