EXAM 3
MATH 161

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
By writing your name on the line below, you indicate that you have read and understand these directions.

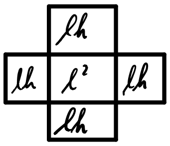
Name: Solutions

Problem	Points Earned	Points Possible
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: November 28, 2018.

Problems

1 (10 Points)**.** A box with a square base and open top must have a volume of 32 cm^3. Find the dimensions of the box that minimize the amount of material used.



$V = \ell^2 h = 32 \Rightarrow h = 32/\ell^2$

$A = \ell^2 + 4\ell h = \ell^2 + 4\ell(32/\ell^2)$

$= \ell^2 + \frac{128}{\ell}$

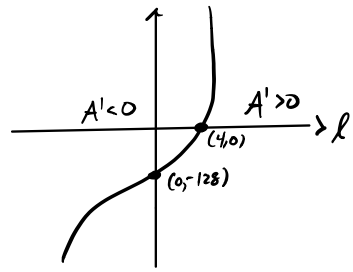
$A' = 2\ell - \frac{128}{\ell^2} = \frac{2\ell^3 - 128}{\ell^2} = 0$

$\iff 2\ell^3 - 128 = 0$

$\iff \ell^3 = \frac{128}{2} = 64 = 2^6$

$\iff \ell = (2^6)^{1/3} = 2^{6/3} = 2^2 = 4$

Looking at the graph of $2\ell^3 - 128$



it's clear from the First Derivative Test that $\ell = 4$ is a minimum, so

$$\boxed{\ell = 4 \quad \text{and} \quad h = \frac{32}{4^2} = \frac{32}{16} = 2.}$$

2 (10 Points)**.** Let

$$f(x) = \int_0^x \cos(t)\,\mathrm{d}t.$$

Use Part I of the Fundamental Theorem of Calculus to evaluate

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x^2) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x^2} \cos(t)\,\mathrm{d}t.$$

By Part I of the FTC, $f'(x) = \cos(x)$, so

$$\boxed{\frac{d}{dx} \int_0^{x^2} \cos(t)\,dt = f'(x^2)\frac{d}{dx}(x^2) = 2x\cos(x^2).}$$

3 (10 Points). Evaluate the integral $\int_0^2 3x^2\,dx$ using the limit definition of the integral and the identity

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}.$$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = 0 + i\Delta x = \frac{2i}{n}$, $f(x) = x^2$

$\underline{n>0}$ $\quad R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{8}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right) = \frac{4}{3}\left(\frac{2n^3+3n^2+n}{n^3}\right) = \frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$\int_0^2 3x^2\,dx = 3\int_0^2 x^2\,dx = 3\lim_{n\to\infty} R_n = 3\lim_{n\to\infty} \frac{4}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = 3\left(4/3\right)\left(2+0+0\right) = \boxed{8}$

4 (10 Points). Use the Fundamental Theorem of Calculus Part II to check your answer to Problem 3.

$\int_0^2 3x^2\,dx = 3\int_0^2 x^2\,dx = 3\left(\frac{1}{3}\right) x^3\Big|_0^2 = 2^3 - 0^3 = \boxed{8}$

5 (10 Points). Evaluate the indefinite integral

$$\int \frac{\sin(x)}{\cos^2(x)}\,\mathrm{d}x.$$

Method 1

$$\int \frac{\sin(x)}{\cos^2(x)}dx = \int \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}dx = \int \sec(x)\tan(x)dx = \boxed{\sec(x)+C.}$$

Method 2

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x) \Rightarrow -du = \sin(x)dx$

$$\int \frac{\sin(x)}{\cos^2(x)}dx = \int \frac{-du}{u^2} = -\int u^{-2}du = -(-u^{-1})+C = \frac{1}{u}+C = \frac{1}{\cos(x)}+C = \boxed{\sec(x)+C}$$

6 (10 Points). Evaluate the indefinite integral

$$\int \csc(\theta)\big(\sin(\theta)-\csc(\theta)\big)\,\mathrm{d}\theta$$

$$\begin{aligned}\int \csc(\theta)(\sin(\theta)-\csc(\theta))d\theta &= \int \csc(\theta)\sin(\theta)d\theta - \int \csc^2(\theta)d\theta\\ &= \int \frac{1}{\sin(\theta)}\sin(\theta)d\theta - \int \csc^2(\theta)d\theta\\ &= \int d\theta - \int \csc^2(\theta)d\theta\\ &= \theta - (-\cot(\theta)) + C\\ &= \boxed{\theta + \cot(\theta) + C.}\end{aligned}$$

7 (10 Points). Evaluate the indefinite integral

$$\int 2x\sqrt{x-5}\,\mathrm{d}x$$

Let $u = x-5$, so $du = dx$ and $x = u+5$. Then

$$\int 2x\sqrt{x-5}\,dx = 2\int (u+5)\sqrt{u}\,du = 2\left[\int u\sqrt{u}\,du + 5\int \sqrt{u}\,du\right]$$

$$= 2\int u^{3/2}du + 10\int u^{1/2}du$$

$$= 2\left(\frac{2}{5}\right)u^{5/2} + 10\left(\frac{2}{3}\right)u^{3/2} + C$$

$$= \boxed{\frac{4(x-5)^{5/2}}{5} + \frac{20}{3}(x-5)^{3/2} + C.}$$

Check: $\frac{d}{dx}\left(\frac{4(x-5)^{5/2}}{5} + \frac{20}{3}(x-5)^{3/2} + C\right) = \frac{4}{5}\left(\frac{5}{2}\right)(x-5)^{3/2} + \frac{20}{3}\left(\frac{3}{2}\right)(x-5)^{1/2} = 2(x-5)^{3/2} + 10(x-5)^{1/2}$

$$= 2\left((x-5)\sqrt{x-5} + 5\sqrt{x-5}\right) = 2(x-5+5)\sqrt{x-5} = 2x\sqrt{x-5} \checkmark$$

8 (10 Points). Evaluate the definite integral

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$$

Let $u = x^2$, so

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$$

$u(0) = 0$, $u(\sqrt{\pi}) = (\sqrt{\pi})^2 = \pi$.

$$\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \int_0^{\pi} \frac{1}{2}\sin(u)\,du = -\frac{1}{2}\cos(u)\Big|_0^{\pi}$$

$$= -\frac{1}{2}\left[\cos(\pi) - \cos(0)\right]$$

$$= -\frac{1}{2}\left[-1-1\right]$$

$$= -\frac{1}{2}(-2)$$

$$= \boxed{1}$$

9 (10 Points). Given $f'(x) = 12x^2 + 6x - 4$ and $f(1) = 1$, find $f(x)$.

$$f(x) = \int f'(x)dx = \int 12x^2+6x-4\,dx = 12\int x^2 dx + 6\int x\,dx - 4\int dx = 12\left(\tfrac{1}{3}x^3\right) + 6\left(\tfrac{1}{2}x^2\right) - 4x + C$$

$$= 4x^3 + 3x^2 - 4x + C.$$

$$1 = f(1) = 4(1)^3 + 3(1)^2 - 4(1) + C = 4+3-4+C = 3+C$$

$$\Rightarrow C = 1-3 = -2$$

Therefore

$$\boxed{f(x) = 4x^3+3x^2-4x-2}$$

10 (10 Points). Assume that f is an even function. Given

$$\int_{-1}^{3} f(x)\,dx = 3 \text{ and } \int_{1}^{3} f(x)\,dx = 1$$

find

$$\int_{0}^{1} f(x)\,dx.$$

Because f is even, $\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$, so

$$\int_{-1}^{3} f(x)dx = \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)dx = 2\int_{0}^{1} f(x)dx + \int_{1}^{3} f(x)dx$$

$$\Rightarrow \int_{0}^{1} f(x)dx = \frac{1}{2}\left[\int_{-1}^{3} f(x)dx - \int_{1}^{3} f(x)dx\right] = \frac{1}{2}[3-1] = \frac{1}{2}(2) = \boxed{1}$$