EXAM 3

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period. It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.** You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Date: November 13, 2019.

Derivatives

1. Use the **limit definition**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to compute the derivative of

$$f(x) = \sqrt{x}.$$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

Product and Quotient Rules

2. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\cos(x)\right].$$

$$\frac{d}{dx}\left(\sin(x)\cos(x)\right) = \frac{d}{dx}(\sin(x))\cos(x) + \sin(x)\frac{d}{dx}\cos(x)$$
$$= \cos(x)\cos(x) + \sin(x)(-\sin(x))$$
$$= \boxed{\cos^2(x) - \sin^2(x)}$$

3. Compute
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{3x+2}{\cos(x)}\right].$$

$$\frac{d}{dx}\left[\frac{3x+2}{\cos(x)}\right] = \frac{\frac{d}{dx}(3x+2)\cos(x) - (3x+2)\frac{d}{dx}\cos(x)}{\cos^2(x)}$$
$$= \frac{3\cos(x) - (3x+2)(-\sin(x))}{\cos^2(x)}$$
$$= \boxed{\frac{3\cos(x) + (3x+2)\sin(x)}{\cos^2(x)}}$$

Chain Rule

4. Compute

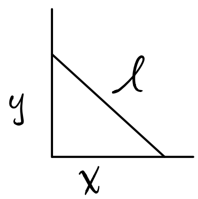
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sec\left(x^2\right)\right].$$

$$\frac{d}{dx}\sec(x^2) = \sec(x^2)\tan(x^2)\frac{d}{dx}(x^2)$$

$$= \boxed{2x\sec(x^2)\tan(x^2)}$$

Implicit Differentiation and Related Rates

5. The top of a ladder slides down a vertical wall at a rate of 3 meters/second. At the moment when the top of the ladder is 4 meters from the ground, it slides away from the wall at a rate of 4 meters/second. How long is the ladder?



$$\frac{dy}{dt} = -3, \quad \frac{dx}{dt} = 4, \quad \frac{d\ell}{dt} = 0$$

$$x^2 + y^2 = \ell^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt} = 0$$

$$\Rightarrow x = \frac{-2y\frac{dy}{dt}}{2\frac{dx}{dt}} = \frac{-y\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4(-3)}{4} = 3$$

$$\ell = \sqrt{4^2+3^2} = \sqrt{25} = \boxed{5\text{m}}$$

Derivatives and Shape

Use the function

$$f(x) = x^3 + 9x^2$$

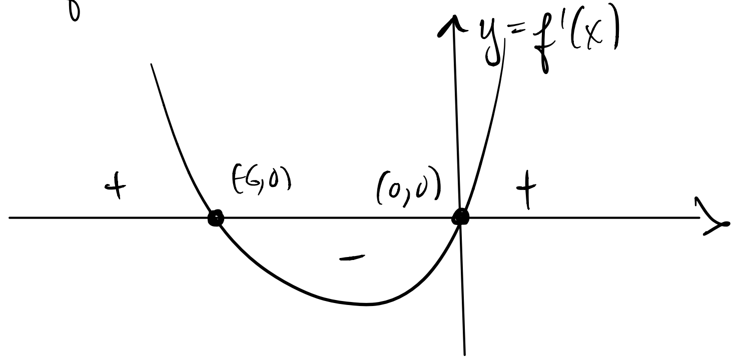
to answer the following questions.

6. Find the intervals where the function is increasing and decreasing. Write NONE if not applicable.

Increasing: $(-\infty, -6) \cup (0, \infty)$

Decreasing: $(-6, 0)$

$$f'(x) = 3x^2 + 18x = 3x(x+6)$$



7. State the local maximum and local minimum value(s). Write NONE if not applicable.

Local maximum value(s): $(-6, 108)$

Local minimum value(s): $(0, 0)$

$$f(-6) = (-6)^3 + 9(-6)^2$$
$$= 9(6)^2 - 6^3$$
$$= 6^2(9-6)$$
$$= 36(3) = 90 + 18 = 108$$

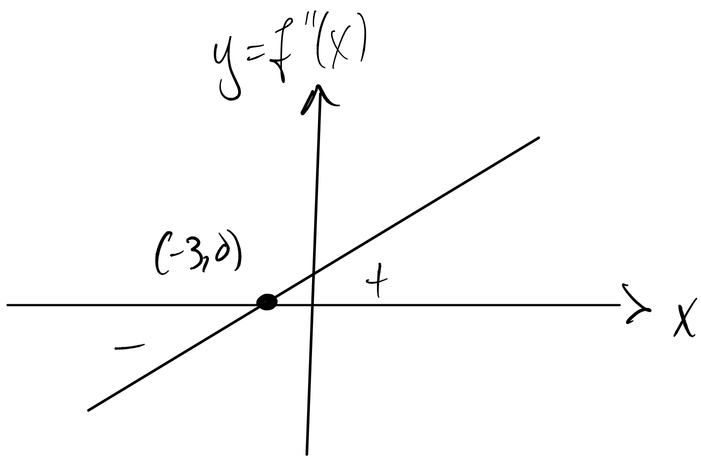
8. Find the intervals on which the function is concave up and concave down. State the inflection points. Write NONE if not applicable.

Concave Up: $(-3, \infty)$

Concave Down: $(-\infty, -3)$

Inflection Points: $(-3, 54)$

$$f''(x) = 6x + 18 = 6(x+3)$$



$$f(-3) = (-3)^3 + 9(-3)^2$$
$$= -9(3) + 9(9)$$
$$= 9(9-3)$$
$$= 9(6)$$
$$= 54$$

Asymptotes

9. Find the asymptotes of

$$f(x) = \frac{2x^2 + 2x - 12}{x^2 - 9}$$

Write NONE if there are none.

Horizontal: $y = 2$

Vertical: $x = 3$

$$\lim_{x \to \infty} \frac{2x^2 + 2x - 12}{x^2 - 9} = \lim_{x \to \infty} \frac{x^2}{x^2}\left(\frac{2 + 2/x - 12/x^2}{1 - 9/x^2}\right)$$

$$= \lim_{x \to \infty} \frac{2 + 2/x - 12/x^2}{1 - 9/x^2} = \frac{2 + 0 - 0}{1 - 0} = 2$$

$$\lim_{x \to -3} \frac{2x^2 + 2x - 12}{x^2 - 9} = \lim_{x \to -3} \frac{2(x+3)(x-2)}{(x+3)(x-3)} = \lim_{x \to -3} \frac{2(x-2)}{x-3}$$

$$= \frac{2(-3-2)}{-3-3} = \frac{2(-5)}{-6} = \frac{5}{3}$$

$$\lim_{x \to 3^+} \frac{2x^2 + 2x - 12}{x^2 - 9} = \lim_{x \to 3^+} \frac{2(x-2)}{x-3} = \infty$$

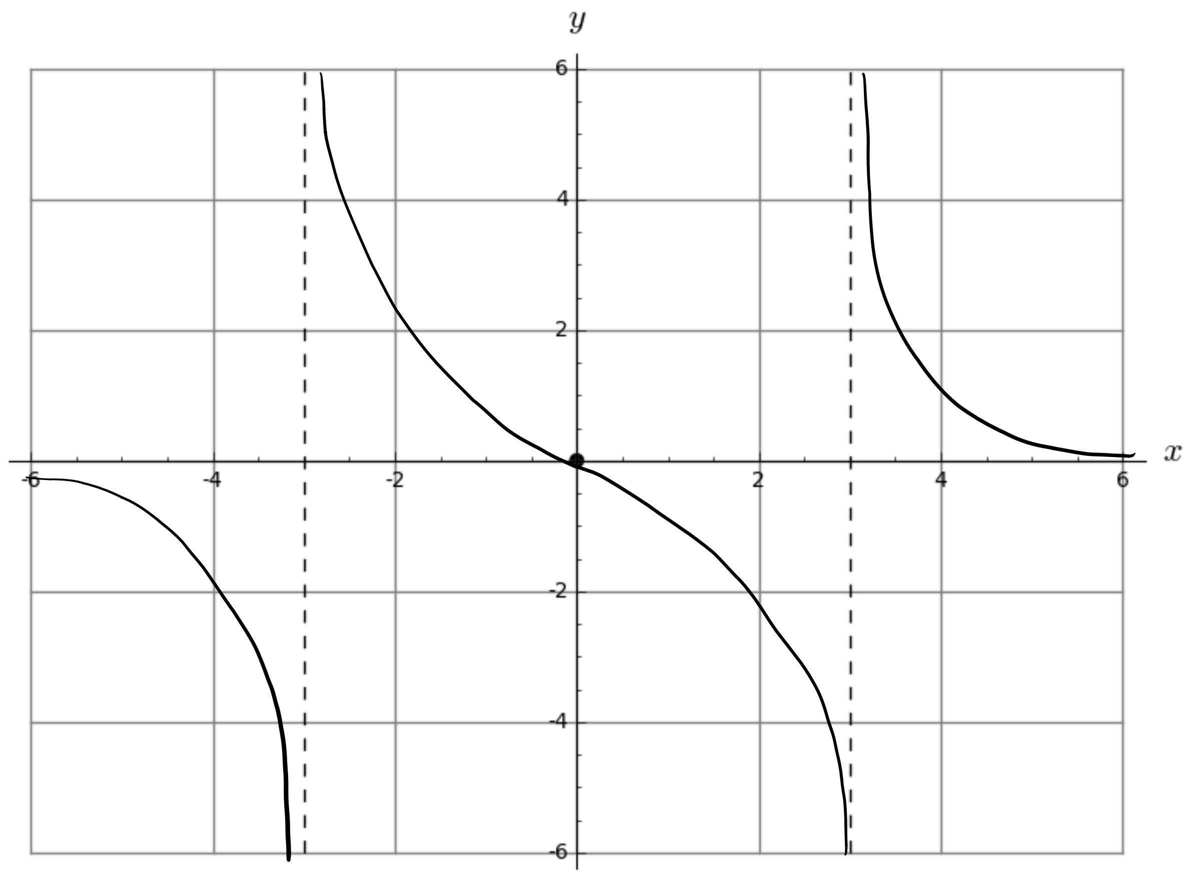
$$\lim_{x \to 3^-} \frac{2x^2 + 2x - 12}{x^2 - 9} = \lim_{x \to 3^-} \frac{2(x-2)}{x-3} = -\infty.$$

Curve Sketching

10. You are given the following information about a function, f:

- x-intercept(s): $(0, 0)$,
- y-intercept: $(0, 0)$,
- asymptotes: $x = -3$, $x = 3$, $y = 0$,
- critical point(s): None,
- decreasing: $(-\infty, \infty)$
- concave up: $(-3, 0) \cup (3, \infty)$, and
- concave down: $(-\infty, -3) \cup (0, 3)$

Use the information above to sketch the curve below. The point $(0, 0)$ has been plotted for you.



Closed Interval Method and Optimization

11. Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x$ on $[-1, 4]$.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$$

$$f(x) = x(x^2 - 6x + 9) = x(x-3)^2$$

$$f(-1) = -1(-1-3)^2 = -(-4)^2 = -16$$

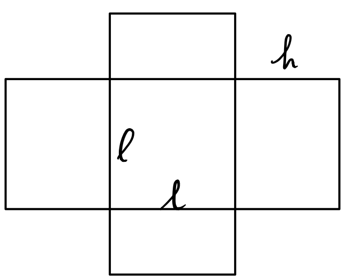
$$f(1) = 1(1-3)^2 = (-2)^2 = 4$$

$$f(3) = 3(3-3)^2 = 0$$

$$f(4) = 4(4-3)^2 = 4(1)^2 = 4$$

$$\boxed{\text{Max}: (1,4) \text{ and } (4,4) \quad \text{Min}: (-1,-16)}$$

12. A box with a square base and no top must have a volume of $4\,\text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.



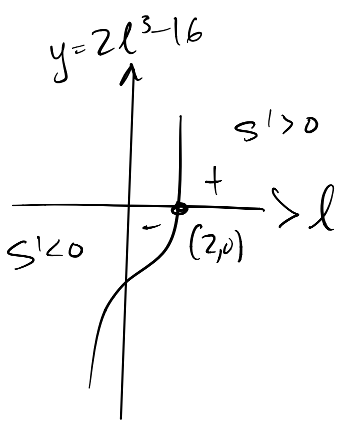
$$V = l^2 h = 4 \Rightarrow h = 4/l^2$$

$$S = 4lh + l^2$$

$$= 4l\left(\frac{4}{l^2}\right) + l^2$$

$$= \frac{16}{l} + l^2$$

$$S' = \frac{-16}{l^2} + 2l = \frac{-16 + 2l^3}{l^2} = 0$$



$$\Rightarrow -16 + 2l^3 = 0$$

$$\Rightarrow 2l^3 = 16$$

$$\Rightarrow l^3 = \frac{16}{2} = 8 \Rightarrow l = 2,\ h = \frac{4}{2^2} = 1$$

$$\boxed{l = 2,\ h = 1}$$

Integration

13. Use right endpoints and the identity

$$\cancel{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

to compute the definite integral

$$\int_0^1 3x^2 \, \mathrm{d}x$$

by the **definition**.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \quad x_i = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$\int_0^1 3x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3i^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{3}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{3}{6} \frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{2n^2+3n+1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{1}{2}(2+0+0) = \boxed{1}$$

Fundamental Theorem of Calculus

Evaluate the following integrals.

14. $\int \left(x^2 - x + \cos(x) - \sec^2(x)\right) \mathrm{d}x.$

$$\int (x^2 - x + \cos(x) - \sec^2(x))\,dx = \int x^2 dx - \int x dx + \int \cos(x) dx - \int \sec^2(x) dx$$

$$= \boxed{\frac{1}{3}x^3 - \frac{1}{2}x^2 + \sin(x) - \tan(x) + C}$$

15. $\int_0^\pi \sin(x)\,\mathrm{d}x$

$$\int_0^\pi \sin(x)dx = -\cos(x)\Big|_0^\pi = -(\cos(\pi) - \cos(0))$$

$$= -(-1-1)$$

$$= -(-2)$$

$$= \boxed{2}$$

Substitution

16. Evaluate the indefinite integral $\int 2\sin(x)\cos(x)\,dx$

$u = \sin(x)$

$du = \cos(x)dx$

$$\int 2\sin(x)\cos(x)\,dx = 2\int u\,du = 2\left(\tfrac{1}{2}\right)u^2 + C = \boxed{\sin^2(x) + C}$$

or

$u = \cos(x)$

$du = -\sin(x)dx$

$\Rightarrow -du = \sin(x)dx$

$$\int 2\sin(x)\cos(x)dx = -2\int u\,du = -2\left(\tfrac{1}{2}\right)u^2 + C$$

$$= \boxed{-\cos^2(x) + C}$$

or

$$\int 2\sin(x)\cos(x) = \int \sin(2x)dx = \tfrac{1}{2}\int \sin(u)du = \boxed{-\tfrac{1}{2}\cos(2x) + C}$$

$u = 2x$, $\tfrac{1}{2}du = dx$

17. Evaluate the definite integral $\int_0^2 \frac{20x}{(x^2+1)^2}\,dx$

$u = x^2+1$ $\quad u(2) = 2^2+1 = 4+1 = 5$

$du = 2x\,dx$ $\quad u(0) = 0^2+1 = 1$

$\tfrac{1}{2}du = x\,dx$

$$\int_0^2 \frac{20x}{(x^2+1)^2}dx = \int_1^5 \frac{10\,du}{u^2} = 10\int_1^5 u^{-2}du$$

$$= 10\left(\frac{1}{-1}\right)u^{-1}\Big|_1^5$$

$$= -10\left(\frac{1}{5} - \frac{1}{1}\right)$$

$$= -2 + 10$$

$$= \boxed{8}$$