EXAM 3

BLAKE FARMAN

Lafayette College

Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.

Name: Solutions

Date: November 22, 2019.

Inverse Trig Functions

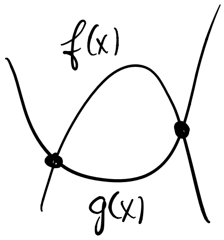
1. Compute
$$\int \frac{2x}{1+x^4}\,\mathrm{d}x$$

$u = x^2$
$du = 2x\,dx$

$$\int \frac{2x\,dx}{1+(x^2)^2} = \int \frac{du}{1+u^2} = \arctan(u) + C$$
$$= \boxed{\arctan(x^2) + C}$$

Area and Volumes

2. Compute the area of the region bounded by $f(x) = -x^2+2x+8$ and $g(x) = 2x^2+2x-4$.



$$0 = f(x) - g(x) = -x^2+2x+8 - 2x^2 - 2x + 4 = -3x^2+12$$

$$\Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\int_{-2}^{2} (f(x)-g(x))dx = \int_{-2}^{2} (-3x^2+12)dx$$

$$= 2\int_0^2 (-3x^2+12)\,dx$$

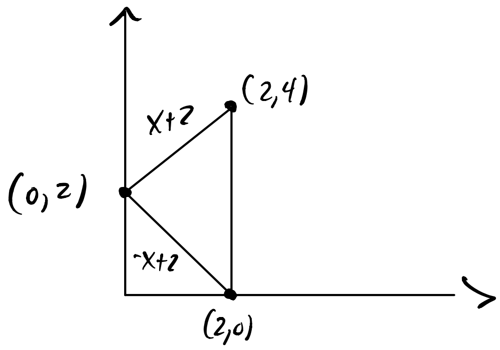
$$= 2\left(-x^3\Big|_0^2 + 12x\Big|_0^2\right)$$

$$= 2\left(-(8-0) + 12(2-0)\right)$$

$$= 2(-8+24)$$

$$= \boxed{32}$$

3. Compute the volume of the solid obtained by revolving the region in the first quadrant bounded by the lines $x = 2$, $y = x + 2$, and $y = 2 - x$ about the x-axis.



Area of Cross-Section:

$$A(x) = \pi(x+2)^2 - \pi(2-x)^2$$

$$= \pi\left[(x^2+4x+4) - (4-4x+x^2)\right]$$

$$= \pi\left[x^2+4x+4-4+4x-x^2\right]$$

$$= 8\pi x$$

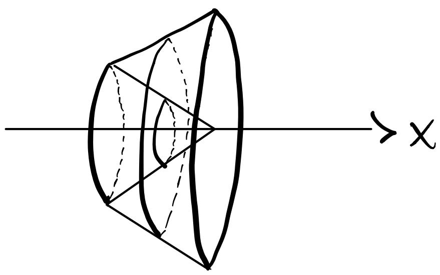
$$V = \int_0^2 8\pi x\,dx$$

$$= 4\pi x^2\Big|_0^2$$

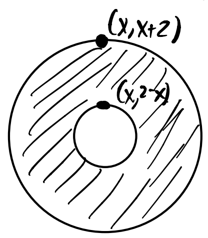
$$= 4\pi(4-0)$$

$$= \boxed{16\pi}$$

Sketch of Solid:



Cross-Section



Integration by Parts

4. Compute
$$\int x^2 \cos(x)\,\mathrm{d}x$$

$u = x^2 \qquad v = \sin(x)$

$du = 2x \qquad du = \cos(x)dx$

$u = x \qquad v = -\cos(x)$

$du = dx \qquad du = \sin(x)dx$

$$\int x^2\cos(x)dx = x^2\sin(x) - 2\int x\sin(x)dx$$
$$= x^2\sin(x) - 2\left(-x\cos(x) - \int -\cos(x)dx\right)$$
$$= x^2\sin(x) + 2x\cos(x) - 2\int \cos(x)dx$$
$$= \boxed{x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C}$$

Check: $\frac{d}{dx}\left(x^2\sin(x) + 2x\cos(x) - 2\sin(x)\right)$
$$= 2x\sin(x) + x^2\cos(x) + 2\cos(x) - 2x\sin(x) - 2\cos(x)$$
$$= x^2\cos(x) \checkmark$$

Partial Fractions

5. Compute

$$\int \frac{16x+22}{(2x+3)(4x+5)}\,\mathrm{d}x$$

$$\frac{16x+22}{(2x+3)(4x+5)} = \frac{A}{2x+3} + \frac{B}{4x+5}$$

$$\Rightarrow 16x+22 = A(4x+5) + B(2x+3)$$

$\underline{x=-\frac{5}{4}}$

$$\Rightarrow 16\left(-\tfrac{5}{4}\right)+22 = -20+22 = 2 = A(0) + B\left(2\left(\tfrac{-5}{4}\right)+3\right) = B\left(-\tfrac{5}{2}+\tfrac{6}{2}\right) = \tfrac{B}{2}$$

$$\Rightarrow B = 4$$

$\underline{x=-\frac{3}{2}}$

$$\Rightarrow 16(-3/2)+22 = -24+22 = -2 = A\left(4\left(-\tfrac{3}{2}\right)+5\right) + B(0)$$
$$= A(-6+5) = -A$$

$$\Rightarrow A = 2.$$

$$\int \frac{16x+22}{(2x+3)(4x+5)}\,dx = \int \frac{2}{2x+3}\,dx + \int \frac{4}{4x+5}\,dx$$
$$= 2\,\frac{\ln|2x+3|}{2} + 4\,\frac{\ln|4x+5|}{4} + C$$
$$= \boxed{\ln|2x+3| + \ln|4x+5| + C}$$

Check: $\frac{d}{dx}\left(\ln|2x+3| + \ln|4x+5|\right) = \frac{2}{2x+3} + \frac{4}{4x+5} = \frac{(8x+10)+(8x+12)}{(2x+3)(4x+5)}$

$$= \frac{16x+22}{(2x+3)(4x+5)} \checkmark$$

Approximate Integration

Use the function $f(x) = x^3 - x + 1$ to answer Problems 6 and 7.

6. Use the inequality

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ on $[a, b]$, to find the number of intervals needed to estimate

$$\int_0^4 f(x)\,\mathrm{d}x$$

using the Midpoint Rule with an error less than 10^{-2}.

$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

$|6x| \leq 6(4) = 24$ on $[0,4]$

$|E_M| \leq \frac{24(4-0)^3}{24n^2} = \frac{4^3}{n^2} < \frac{1}{10^2}$

$\Rightarrow 4^3 10^2 < n^2$

$\Rightarrow \sqrt{4^3 10^2} = 2^3 \cdot 10 = \boxed{80 < n}$

7. Use the Midpoint Rule

$$M_n = \sum_{i=1}^{n} f(\overline{x_i})\Delta x, \quad \overline{x_i} = \frac{x_i + x_{i-1}}{2}$$

with $n = 2$ intervals to estimate the value of the integral

$$\int_0^4 f(x)\,\mathrm{d}x.$$

$f(x) = x^3 - x + 1$

$\Delta x = \frac{4-0}{2} = \frac{4}{2} = 2$

$x_0 = 0, \; x_1 = 2, \; x_2 = 4$

$\overline{x_1} = \frac{2+0}{2} = 1, \; \overline{x_2} = \frac{2+4}{2} = 3$

$f(1) = 1^3 - 1 + 1 = 1, \; f(3) = 27 - 3 + 1 = 25$

$M_2 = [f(1) + f(3)]\,2 = 26 \cdot 2 = \boxed{52}$

Improper Integrals

8. Compute the improper integral

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,\mathrm{d}x$$

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t \to -\infty} \left[\arctan(0) - \arctan(t)\right]$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

Differential Equations

9. Solve the inital value problem
$$y' = \frac{3x^2}{2y}, \quad y(2) = 2.$$

$$\int 2y\,dy = \int 3x^2\,dx$$

$$\Rightarrow y^2 = x^3 + C$$

$$2^2 = 2^3 + C \Rightarrow 4 = 8 + C \Rightarrow C = 4 - 8 = -4$$

So

$$\boxed{y = \sqrt{x^3 - 4}}$$

Check: $y' = \frac{1}{2\sqrt{x^3-4}}(3x^2) = \frac{3x^2}{2y}$ ✓

Parametric Curves

10. Sketch the curve parametrized by

$$x = t + 2, \quad y = t^3 - t, \quad -1 \le t \le \cancel{-1}\ \underline{1}$$

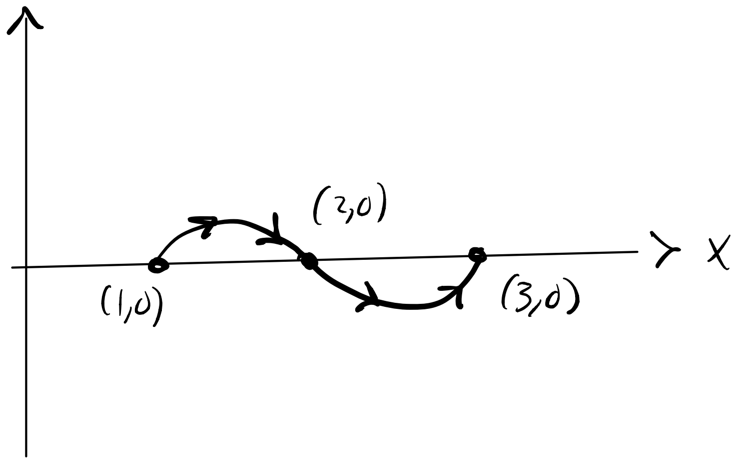
and indicate with an arrow the direction in which the curve is traced as t increases.

$$y = t(t^2-1) = t(t+1)(t-1) = (x-2)(x-1)(x-3)$$

on the interval

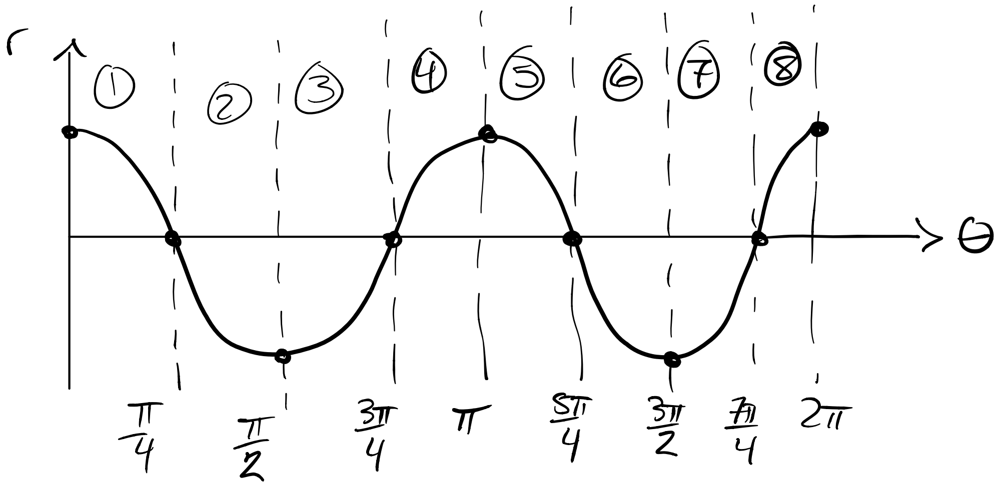
$$-1 \le t = x-2 \le 1 \Rightarrow -1+2 \le x \le 1+2$$

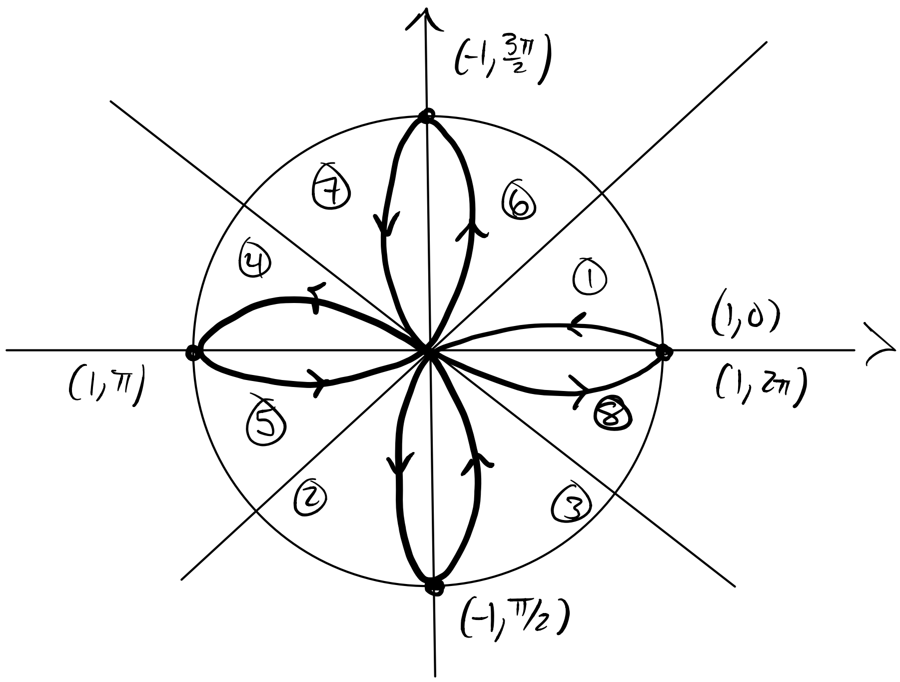
$$\Rightarrow 1 \le x \le 3$$



Polar Coordinates

11. Sketch the rose $r = \cos(2\theta)$, $0 \le \theta \le 2\pi$. Label the tips of the petals and draw arrows to indicate the direction in which the curve is traced.





Sequences

12. Find a forumla for the general term of the sequence $\{a_n\}_{n=1}^{\infty}$ assuming the pattern continues, then compute the limit of the sequence.

$$\left\{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \ldots\right\}$$

$$a_n = \sqrt[n]{n} = n^{1/n}$$

$$\lim_{n\to\infty} \ln\left(n^{1/n}\right) = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So

$$\lim_{n\to\infty} \sqrt[n]{n} = e^{\lim_{n\to\infty} \ln(n^{1/n})}$$

$$= e^{0}$$

$$= \boxed{1}$$

Geometric Series

13. Assuming that the pattern continues, compute the sum of the series

$$12 + 6 + 3 + \frac{3}{2} + \frac{3}{4} + \dots$$

$a = 12, \quad r = \frac{6}{12} = \frac{1}{2}$

$$12 + 6 + 3 + \frac{3}{2} + \frac{3}{4} + \dots = \sum_{n=1}^{\infty} 12\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{12}{1 - \frac{1}{2}}$$

$$= \frac{12}{\frac{1}{2}}$$

$$= \boxed{24}$$

The Integral Test

14. Use the **Integral Test** to determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

$\left[\text{Hint: } \dfrac{1}{x^2+x} = \dfrac{x+1-x}{x^2+x}\right]$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^t \left[\frac{x+1}{x(x+1)} - \frac{x}{x(x+1)}\right] dx$$

$$= \lim_{t\to\infty} \left[\int_1^t \frac{1}{x}\,dx - \int_1^t \frac{1}{x+1}\,dx\right]$$

$$= \lim_{t\to\infty} \left[\ln|x|\Big|_1^t - \ln|x+1|\Big|_1^t\right]$$

$$= \lim_{t\to\infty} \left[(\ln(t) - \ln(1)) - (\ln|t+1| - \ln(2))\right]$$

$$= \lim_{t\to\infty} \left[\ln\left(\frac{t}{t+1}\right) + \ln(2)\right]$$

$$= \ln(1) + \ln(2)$$

$$= \ln(2) < \infty$$

So

$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ $\boxed{\text{converges}}$ because $\int_1^{\infty} \frac{1}{x^2+x}\,dx$ converges.

The Comparison Tests

15. Use either the Comparison Test or the Limit Comparison Test to decide whether the following series converges or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{n+1}{\sqrt[3]{8n^7 - 2n + 1}}$$

$$a_n = \frac{n+1}{\sqrt[3]{8n^7-2n+1}} \qquad b_n = \frac{n}{\sqrt[3]{8n^7}} = \frac{n^{3/3}}{2n^{7/3}} = \frac{1}{2n^{4/3}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{\sqrt[3]{8n^7-2n+1}}\left(\frac{2n^{4/3}}{1}\right)$$

$$= \lim_{n\to\infty} \frac{2n^{7/3}+2n^{4/3}}{(8n^7-2n+1)^{1/3}}$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[3]{8n^7}}{\sqrt{8n^7-2n+1}} + \frac{\sqrt[3]{8n^4}}{\sqrt[3]{8n^7-2n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt[3]{\frac{8n^7}{8n^7-2n+1}} + \sqrt[3]{\frac{8n^4}{8n^7-2n+1}}\right)$$

$$= \sqrt[3]{\frac{8}{8}} + \sqrt[3]{0}$$

$$= 1 > 0$$

So both series $\boxed{\text{Converge}}$ ($\sum b_n$ is a p-series, $p>1$).

Alternating Series

16. Decide whether the following series converges or diverges. Justify your answer.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n}.$$

$$\frac{d}{dx}\frac{x}{e^x} = \frac{d}{dx}xe^{-x} = e^{-x} + x(-e^{-x})$$

$$= \frac{1}{e^x} - \frac{x}{e^x}$$

$$= \frac{1-x}{e^x} < 0 \quad \text{for } x > 1$$

so $b_{n+1} \leq b_n$ for $n \geq 2$ and

$$\lim_{n\to\infty}\frac{n}{e^n} \overset{L'H}{=} \lim_{n\to\infty}\frac{1}{e^n} = 0$$

Therefore $\sum_{n=1}^{\infty}(-1)^n\frac{n}{e^n}$ $\boxed{\text{converges by the A.S.T.}}$

Ratio and Root Tests

Determine whether the following series converge or diverge.

17. $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$ $\boxed{\text{Diverges by the Ratio Test}}$!

$$\lim_{n\to\infty} \left| \frac{(n+1)5^{2n+2}}{10^{n+2}} \frac{10^{n+1}}{n5^{2n}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \frac{5^{2n+2}}{5^{2n}} \frac{10^{n+1}}{10^{n+2}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \frac{25}{10} = \frac{25}{10} > 1$$

18. $\sum_{n=1}^{\infty} \left(\frac{n}{\ln(n)}\right)^n$ $\boxed{\text{Diverges by the Root Test}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{\ln(n)}\right|^n} = \lim_{n\to\infty} \frac{n}{\ln(n)}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} n = \infty > 1.$$

Power Series

19. Determine the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{\sqrt{n}}x^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}4^{n+1}x^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(-1)^n4^nx^n}\right| = \lim_{n\to\infty}\frac{4^{n+1}}{4^n}\frac{\sqrt{n}}{\sqrt{n+1}}\frac{|x|^{n+1}}{|x|^n}$$

$$= \lim_{n\to\infty} 4|x|\sqrt{\frac{n}{n+1}} = 4|x|\sqrt{1} = 4|x| < 1$$

$$\Rightarrow |x| < \frac{1}{4}$$

$\underline{x = \frac{-1}{4}}$ $\quad \sum_{n=1}^{\infty}\frac{(-1)^n4^n}{\sqrt{n}}\left(\frac{-1}{4}\right)^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series

$\underline{x = \frac{1}{4}}$ $\quad \sum_{n=1}^{\infty}\frac{(-1)^n4^n}{\sqrt{n}}\left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the A.S.T.:

$\sqrt{n}$ is increasing, so $b_n = \frac{1}{\sqrt{n}}$ is decreasing and

$\lim_{n\to\infty} b_n = 0.$

Radius of Convergence: $\boxed{\frac{1}{4}}$

Interval of Convergence: $\boxed{\left(-\frac{1}{4}, \frac{1}{4}\right]}$