# EXAM 3

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Answer the questions in the spaces provided on the question sheets and turn them in at the end of the exam period.
It is advised, although not required, that you check your answers. All supporting work is required. Unsupported or otherwise mysterious answers will **not receive credit.**
You may **not** use a calculator or any other electronic device, including cell phones, smart watches, etc.
By writing your name on the line below, you indicate that you have read and understand these directions.

Name: Solutions

| Problem | Points Earned | Points Possible |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 20 |
| Bonus | | 5 |
| Total | | 100 |

*Date*: May 1, 2019.

## Series

For each of the problems in this section, determine whether the given series converges or diverges. Clearly indicate the test that you use.

**1.** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ $\quad b_n = \frac{n^2}{n^3+4}$

- Identify that the A.S.T. needs to be used (5 points)
- Show $\{b_n\}$ is decreasing (5 points):

$f(x) = \frac{x^2}{x^3+4}$, $f'(x) = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2} = \frac{2x^4-3x^4+8x}{(x+4)^2} = \frac{-x(x^3+8)}{(x+4)^2} < 0$

on $(0,\infty)$ so

$$b_{n+1} = f(n+1) \leq f(n) = b_n.$$

- Show that

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = 0 \quad (5\text{points})$$

- Conclude the series converges (5 points).

**2.** $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^2}$

Easy Option

- Identify the Ratio Test as an option (5 points)
- Compute (10 points)

$$\lim_{n\to\infty}\left|\frac{(-1)^n 2^{(n+1)}}{(n+1)^2}\frac{n^2}{(-1)^{n-1}2^n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{2n^2}{(n+1)^2} = 2$$

- conclude the series diverges by the Ratio Test because $2>1$. (5 points)

Harder Option

- Identify the $n^{th}$ Term Test for Divergence (5 points)
- Compute

$$\lim_{n\to\infty}\frac{2^n}{n^2} \overset{L'H}{=} \lim_{n\to\infty}\frac{\ln(2)2^n}{2n} \overset{L'H}{=} \lim_{n\to\infty}\frac{(\ln(2))^2 2^n}{2} = \infty \text{ (10 points)}$$

- Argue that, as a result, the limit

$$\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^2}$$

does not exist and thus the series diverges (5 points)

Hardest Option

- Show that $\lim_{n\to\infty}\sqrt[n]{n} = 1$ (15 points)
- Compute $\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty}\frac{2}{\sqrt[n]{n^2}} = \lim_{n\to\infty}\frac{2}{(\sqrt[n]{n})^2} = \frac{2}{1} = 2 > 1$ and conclude the series diverges by the Root Test.

**3.** $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$

- Identify the Root Test (5 points)
- Compute

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3}{n}\right)^n\right|} = \lim_{n\to\infty} \frac{3}{n} = 0 \quad (10 \text{ points})$$

- Conclude that $0<1$, so the series converges by the Root Test (5 points).

## Power Series

**4.** Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$$

- Compute

$$\lim_{n\to\infty} \left| \frac{4^{n+1} x^{2(n+1)}}{(n+1)} \frac{n}{4^n x^{2n}} \right| = \lim_{n\to\infty} 4|x|^2 \left(\frac{n}{n+1}\right) = 4|x|^2 \quad (5 \text{ points})$$

- Solve

$$4|x|^2 < 1 \iff |x|^2 < \frac{1}{4} \iff |x| < \frac{1}{2} = R \iff -\frac{1}{2} < x < \frac{1}{2} \quad (5 \text{ points})$$

- Plug in $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ to get

$$\sum_{n=1}^{\infty} \frac{4^n (\pm\frac{1}{2})^{2n}}{n} = \sum_{n=1}^{\infty} \frac{4^n (\frac{1}{4^n})}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

then observe that this is a divergent series. (5 points)

- Conclude the Interval of Convergence is $(-\frac{1}{2}, \frac{1}{2})$ (5 points).

**5.** Given the Maclaurin series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,\ -1 < x < 1$$

find the Taylor series centered at $a = 1$ for

$$f(x) = \frac{1}{x}.$$

Where is the Taylor series equal to the function?
[Hint: $x = 1 + (x-1)$.]

- Make the substitution:

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty}(-1)^n (x-1)^n \qquad (10 \text{ points})$$

- Observe that this is valid when

$$-1 < x-1 < 1 \iff 0 < x < 2 \quad (10 \text{ points}).$$

**6** (Bonus - 5 Points). Use your answer to Problem 5 to compute the Taylor series for

$$f(x) = \ln(x)$$

centered at $a = 1$. Where is the Taylor series equal to the function?

Observe that $\frac{d}{dx}\frac{(x-1)^{n+1}}{n+1} = \frac{(n+1)(x-1)^n}{(n+1)} = (x-1)^n$ and on $(0,2)$

$$\ln(x) + C = \int \frac{1}{x}\,dx$$

$$= \int\left(\sum_{n=0}^{\infty}(-1)^n(x-1)^n\right)dx$$

$$= \sum_{n=0}^{\infty}\int\left[(-1)^n(x-1)^n\right]dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\int(x-1)^n\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \cdots$$

When $x = 1$ we see that this reduces to

$$\ln(1) + C = 0 + C = C = (1-1) - \frac{(1-1)^2}{2} + \cdots = 0 + 0 + \cdots = 0.$$

Therefore

$$\boxed{\ln(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1} \quad \text{on } (0,2)}$$